§4.4 Compactness and Arzela-Ascoli Theorem

Recall that eg 2.21 in §2.4 shows that closed and bounded subset in $(C[0,1], d_\infty)$ is not compact. We need further condition to ensure the compactness of subsets of $C(X)$.

Def: Let (X, d) be a metric space. A subset $\mathcal{F}$ of $C(X)$ is equicontinuous if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon, \ \forall f \in \mathcal{F} \ \& \ d(x,y) < \delta \ (x, y \in X).$$

Note: Clearly if $\mathcal{F}$ is equicontinuous, then any $\mathcal{F}' \subset \mathcal{F}$ is equicontinuous.

eg: A function f defined on a subset X of $\mathbb{R}^n$ is called Hölder continuous if $\exists \alpha \in (0,1)$ such that

$$(*) \quad |f(x) - f(y)| \leq L|x-y|^\alpha, \quad \forall x, y \in X,$$

for some constant L.

The number α is called the Hölder exponent.

The function is called Lipschitz continuous if $(*)$ holds for $\alpha = 1$.

For a fixed $\alpha \in (0,1]$ & $L>0$, the family

$$\mathcal{F} = \{ f \in C(X) : f \text{ Hölder/Lip. with exponent } \alpha \text{ and } L>0 \}$$

is an equicontinuous family.

Pf: $\forall \varepsilon > 0$, let $\delta > 0$ such that $L\delta^\alpha < \varepsilon$.

Then $\forall f \in \mathcal{F}$, $\forall x,y \in X$ with $|x-y| < \delta$,

$$|f(x) - f(y)| \leq L\,|x-y|^\delta < L\delta^\alpha < \varepsilon. \quad \#$$

Prop 4.7: Let $\mathcal{F}$ be a subset $C(X)$ where X is a convex subset in $\mathbb{R}^n$. Suppose that each function in $\mathcal{F}$ is differentiable and there is a uniform bound on the partial derivatives of those functions in $\mathcal{F}$. Then $\mathcal{F}$ is equicontinuous.

$$\left(\mathcal{F} = \{ f \in C(X) : f \text{ differentiable}, \left\| \left|\frac{\partial f}{\partial x_i}\right| \right\|_\infty \leq M, \forall i \} \right)$$

Pf: $\forall x,y \in X$, X convex $\Rightarrow$ $x + t(y-x) \in X$ $\forall t \in [0,1]$.

Then $f(y) - f(x) = \int_0^1 \frac{d}{dt} f(x+t(y-x))\, dt$

$$= \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x+t(y-x))\,(y_i-x_i)\,dt$$

$$= \sum_{i=1}^n \left(\int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt\right)(y_i-x_i)$$

$$\leq \sqrt{\sum_{i=1}^n \left|\int_0^1 \frac{\partial f}{\partial x_i}(x+t(y-x))\,dt\right|^2}\ \ |y-x|$$

$$\leq \sqrt{n}\,M\,|y-x|,$$ where M = uniform bd. on the partial derivatives

Then by the example, $\mathcal{F}$ is equicontinuous. #

eg44 (Equicontinuous, but not bounded)

Let $\mathcal{A} = \{ y \in [-1,1] : \frac{dy}{dx} = \sin(xy),\ x \in [-1,1] \} \subset C[-1,1]$.

Note: $\mathcal{A} \neq \emptyset$. ($y \equiv 0 \in \mathcal{A}$)

Since $\forall y \in \mathcal{A}$, $|y'| = |\sin(xy)| \leq 1$.

Together with convexity of $[-1,1]$, $\mathcal{A}$ is equicontinuous.

However, $\mathcal{A}$ is not bounded: In fact, we can solve IVP $\begin{cases} y' = \sin(xy) \\ y(0) = y_0 \end{cases}$ for any $y_0 \in \mathbb{R}$.

(as $|\sin(xy_1)-\sin(xy_2)| \le |y_1-y_2|$, Picard-Lindelöf
$\Rightarrow$ solution of IVP.)

$\therefore$ $\|y\|_\infty \ge |y_0|$ can be arbitrary large. #

eg4.5 (Closed & Bounded, but not Equicontinuous)

Let $\mathcal{B}=\{f \in C[0,1] : |f(x)|\le 1, \forall x\in[0,1]\} \subset C[0,1]$.
Then $\mathcal{B}$ is closed and bounded. To show that $\mathcal{B}$ is not equicontinuous, we only need to find a subset of $\mathcal{B}$ which is not equicontinuous:
Let $\{f_n(x)=\sin nx\}_{n=1}^{\infty} \subset \mathcal{B}$. Suppose on the contrary that $\{f_n(x)=\sin nx\}_{n=1}^{\infty}$ is equicontinuous.
Then for $\varepsilon=\frac{1}{2}$, $\exists\, \delta>0$ such that

$\forall\, n\ge 1$, & $x,y\in[0,1]$ with $|x-y|<\delta$, we have
$$|\sin nx - \sin ny| < \frac{1}{2}.$$
However, for any $\delta>0$, if $n>\max\{\frac{\pi}{2\delta}, \frac{\pi}{2}\}$, we have $x=0$ & $y=\frac{\pi}{2n}\in[0,1]$ such that
$|x-y|<\delta$ and

$$\left|\sin n\cdot 0-\sin n\cdot\frac{\pi}{2n}\right|=|0-1|=1>\frac{1}{2}.$$

Which is a contradiction.

$\therefore \{\sin nx\}_{n=1}^{\infty}$ is not equicontinuous.