However, we have

<u>Prop 2.9</u> Every sequentially compact set in a metric space is closed and bounded.

<u>Pf</u>: Let K be a sequentially cpt. set in a metric space (X, d).

If K is empty, then it is clearly closed and bounded,

If $K \neq \emptyset$.

(1) K is closed:

Let $\{x_n\} \subset K$ & $x_n \to x$ in (X, d).

As K is sequentially cpt, $\exists\ x_{n_j} \to y$ for some $y \in K$.

Then by uniqueness of limit $x = y \in K$

$\therefore K$ is closed.

(2) K is bounded.

Suppose not, then $\forall$ fixed point $w \in X$,

$K \not\subset B_n(w)$, $\forall n = 1, 2, 3, \cdots$.

$\Rightarrow \exists\ x_n \in K \setminus B_n(w)$, $\forall n = 1, 2, 3, \cdots$.

Then $\{x_n\} \subset K$, K seq. cpt. $\Rightarrow$

$\exists\ \{x_{n_j}\} \subset \{x_n\}$ s.t. $x_{n_j} \to y$ for some $y \in K$.

Then $n_j \le d(x_{n_j}, w) \le d(x_{n_j}, y) + d(y, w)$

$$\longrightarrow d(y,w).$$

which is a contradiction as $n_j \to +\infty$ as $j \to +\infty$.

$\therefore$ K is bounded. ※

Compactness:

Def: Let E be a subset in a metric space (X, d)

A collection of open sets (finite or infinite)

$\{G_\alpha\}_{\alpha \in A}$ is called an open cover of E

if $E \subset \bigcup_{\alpha \in A} G_\alpha$.

A finite subcover of the open cover $\{G_\alpha\}_{\alpha \in A}$

is a finite subset $\{G_{\alpha_1}, \cdots, G_{\alpha_N}\} \subset \{G_\alpha\}_{\alpha \in A}$

such that $E \subset \bigcup_{k=1}^{N} G_{\alpha_k}$.

Def: Let E be subset of a metric space (X, d).

We call E compact if every open cover of E

admits a finite subcover.

The empty set is defined to be compact.

eg: Closed & bounded subsets of $\mathbb{R}^n$ are compact.

As in the sequentially cpt situation, we have

Prop 2.9' Every compact set in a metric space is closed and bounded.

Pf: Let K be a cpt set in (X, d).

If $K = \phi$, we are done

If $K \neq \phi$:

(1) K is closed:

$\forall y \in X \setminus K$, then $\mathcal{O}_k = \{x : d(x,y) > \frac{1}{k}\}$
$k = 1, 2, 3, \cdots$

is a collection of open sets.

Note that

$\forall x \in K$, then $x \neq y \Rightarrow d(x,y) > 0$.

$\Rightarrow \exists k_0 \geq 1$ s.t. $d(x,y) > \frac{1}{k_0}$.

$\Rightarrow \quad x \in \bigcup_{k=1}^{\infty} \mathcal{O}_k$

$\therefore \{\mathcal{O}_k\}_{k=1}^{\infty}$ is an open cover of K.

Then K cpt $\Rightarrow \exists \{\mathcal{O}_{k_1}, \cdots, \mathcal{O}_{k_N}\}$ s.t.

$K \subset \mathcal{O}_{k_1} \cup \cdots \cup \mathcal{O}_{k_N}$.

Let $k^* = \max\{k_1, \cdots, k_N\}$

Then $\quad \mathcal{O}_{k_j} \subset \mathcal{O}_{k^*}$

$\Rightarrow \quad K \subset \mathcal{O}_{k^*} = \{d(x,y) > \frac{1}{k^*}\}$

$\Rightarrow \quad B_{\frac{1}{k^*}}(y) \subset X \setminus K$.

$\therefore \quad X \setminus K$ is open & hence K is closed.

(2) K is bounded.

Fix any $y \in X$. Then $\{B_n(y)\}_{n=1}^{\infty}$ is a collection of open sets.

Now $\forall x \in K$, $x \in B_n(y)$ for those $n > d(x,y)$

$\Rightarrow \quad K \subset \bigcup_{n=1}^{\infty} B_n(y)$.

$\therefore \{B_n(y)\}_{n=1}^{\infty}$ is an open cover of K.

K cpt $\Rightarrow \exists n_1, \cdots, n_L$ s.t.,

$$K \subset \bigcup_{j=1}^{L} B_{n_j}(y)$$

$\Rightarrow \quad K \subset B_{\max\limits_{j=1,\cdots,L}\{n_j\}}(y)$

$\therefore \quad K$ is bounded. $\#$

Equivalence of sequentially compactness and compactness

Def: A subset E in a metric space (X, d) is said to satisfy the finite intersection property if for all collection of closed sets $\{F_\alpha\}_{\alpha \in A}$ with $\bigcap_{k=1}^{N} (F_{\alpha_k} \cap E) \neq \emptyset$ for all finite subcollection $\{F_{\alpha_k}\}_{k=1}^{N}$, we have $\bigcap_{\alpha \in A} (F_\alpha \cap E) \neq \emptyset$.

Note: in other reference,

A collection of sets $\mathcal{F}$ is said to has the finite intersection property if every finite subcollection of $\mathcal{F}$ has a nonempty intersection.

Our definition:

E satisfies the finite intersection property

$\Longleftrightarrow$ Every collection of closed subsets $\{F_\alpha\}_{\alpha \in A}$ s.t. $\{F_\alpha \cap E\}_{\alpha \in A}$ has finite intersection property, we have $\bigcap_{\alpha \in A} (F_\alpha \cap E) \neq \emptyset$.

<u>Thm 2.12</u> Let E be a subset in a metric space (X,d)

The followings are equivalent :

(a) E is sequentially compact,

(b) E is compact, and

(c) E satisfies the finite intersection property.

[Pf: Omitted. The case for $(X,d)=(\mathbb{R}^n, \text{standard})$ should be proved in Analysis I & II (math 2050/2060). Proof for general topological space will be given in Math 3070 Intro. to topology.]

Properties of cpt set

<u>Prop 2.13</u> Let K be a cpt set and G be an open set in a metric space (X,d). If

$$K \subset G \Rightarrow d(K, \partial G) > 0.$$

Recall: $d(A,B) = \inf\{d(x,y) : x\in A, y\in B\}$.

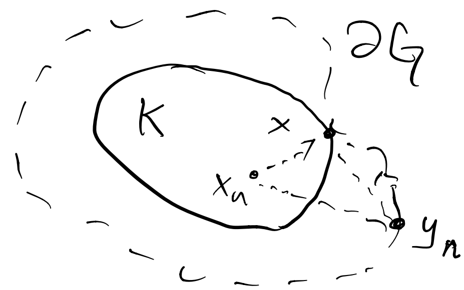
<u>Pf</u>: (Sequentially cpt.)

Suppose on the contrary that

$d(K, \partial G) = 0$.

Then by definition, $\exists\ x_n \in K,\ y_n \in \partial G$ s.t.

$$d(x_n, y_n) \to 0 \text{ as } n \to +\infty$$



As K is (seq.) cpt, $\exists\ x_{n_j} \to x$ for some $x \in K \subset G$.

Now triangle ineq $\Rightarrow$

$$d(y_{n_j}, x) \le d(y_{n_j}, x_{n_j}) + d(x_{n_j}, x)$$
$$\longrightarrow 0 \quad \text{as } j \to +\infty$$

$\therefore\ y_{n_j} \to x$ also.

But $y_{n_j} \in \partial G$ & ∂G is closed $\Rightarrow x \in \partial G$.

However G is open & $x \in K \subset G \Rightarrow x$ is an interior point of G. Therefore $x \notin \partial G$. Contradiction.

$\therefore\ d(K, \partial G) > 0$ ※

(Pf using cptness : Reading Ex!)

<u>Def</u>: Let (X, d) & (Y, ρ) are metric spaces, and $E \subset X$.

Then a map $f : E \to (Y, \rho)$ is called <u>uniformly</u>

continuous on E if

$\forall \varepsilon>0, \exists \delta>0$ such that $\forall x,y \in E$,

$$d(x,y)<\delta \Rightarrow \rho(f(x),f(y))<\varepsilon.$$

Prop 2.14 A continuous map from a cpt set K in (X,d) to (Y,ρ) is uniformly continuous.

Pf: (Sequentially cpt.)

Suppose not,

then $f: K \to (Y,\rho)$ not uniformly cts.

$\Rightarrow \exists \varepsilon_0>0, \forall \delta>0, \exists x,y \in K$ with $d(x,y)<\delta$

and $\rho(f(x),f(y)) \geq \varepsilon_0$.

In particular, for this $\varepsilon_0>0$ consider $\delta=\frac{1}{n}$, $n=1,2,3,\cdots$

Then $\exists x_n, y_n \in K$ with

$$d(x_n,y_n)<\frac{1}{n} \ \& \ \rho(f(x_n),f(y_n)) \geq \varepsilon_0.$$

By (seq) cptness of K, $\exists$ subseq $x_{n_j} \to x$ for some $x \in K$.

As $d(x_{n_j},y_{n_j})<\frac{1}{n_j} \to 0$ as $j \to 0$, we also have

$$y_{n_j} \to x \text{ too}.$$

Then by continuity of f

$$\varepsilon_0 \le \rho(f(x_{n_j}), f(y_{n_j}))$$
$$\le \rho(f(x_{n_j}), f(x)) + \rho(f(x), f(y_{n_j}))$$
$$\longrightarrow 0 \quad \text{as } j \to 0.$$

Contradiction.

$\therefore$ $f: K \to (Y, \rho)$ is uniformly cts on K. #

(Pf using cptness : reading exercise!)

Prop 2.15 : Let E be a cpt set in a metric space (X, d) and $f: (X, d) \to (Y, \rho)$ be cts.

Then $f(E)$ is a cpt set in (Y, ρ).

Pf : (sequentially cpt)

Let $\{y_n\}$ be a seq. in $f(E)$.

Then $\exists \{x_n\} \subset E$ s.t. $f(x_n) = y_n$, $\forall n = 1, 2, 3, \cdots$

By (seq) cptness of E, $\exists$ subseq. $x_{n_j} \to x$ for some $x \in E$.

Then continuity of $f \Rightarrow$

$$y_{n_j} = f(x_{n_j}) \longrightarrow \underset{\in f(E)}{f(x)} \quad \text{as } j \to \infty.$$

$\therefore \{y_{n_j}\}$ is a convergent subseq. of $\{y_n\}$ with limit in $f(E)$.

$\therefore f(E)$ is (seq.) cpt. $\#$

(Pf using cptness - reading ex!)

Ch3 The Contraction Mapping Principle

§3.1 Complete Metric Space

Def: Let (X, d) be a metric space.

(1) A sequence $\{x_n\}$ in (X, d) is a Cauchy sequence if $\forall \varepsilon > 0$, $\exists n_0$ s.t. $d(x_n, x_m) < \varepsilon$, $\forall n, m \geq n_0$.

(2) (X, d) is complete if every Cauchy sequence in (X, d) converges.

(3) A subset E is complete if the induce metric subspace (E, d) is complete.

i.e. every Cauchy sequence in E converges with limit in E.

Note: Convergent sequence is a Cauchy sequence (Ex!)

Prop 3.1 Let (X, d) be a metric space.

(a) If X is complete, then every closed set in X is complete.

(b) Every complete set in X is closed.

(c) Every compact set in X is complete.

Pf: (a) Let (X,d) be complete & E is closed in X.

Then every Cauchy seq. $\{x_n\}$ in E is a Cauchy seq in X. Completeness of $X \Rightarrow \exists x \in X$, s.t. $x_n \to x$. Since E is closed, $x \in E$.

$\therefore$ E is complete.

(b) Let $E \subset X$, & E complete.

Suppose $\{x_n\} \subset E$ with $x_n \to x$ in X.

By note, $\{x_n\}$ is a Cauchy seq. in E

Then completeness of $E \Rightarrow$ $x_n \to z \in E$.

Uniqueness of limit $\Rightarrow$ $x = z \in E$

$\therefore$ E is closed.

(c) Let $K \subset X$ & K cpt.

Let $\{x_n\}$ be a Cauchy seq. in K

Since K cpt, $\exists$ converging subseq $x_{n_j} \to z$ for some $z \in K$.

$\Rightarrow \forall \varepsilon > 0$, $\exists j_0$ s.t. $d(x_{n_j}, z) < \frac{\varepsilon}{2}$, $\forall j \geq j_0$.

$(\Rightarrow n_j \geq n_{j_0})$

On the other hand, $\{x_n\}$ is a Cauchy seq.

$\Rightarrow$ for this $\varepsilon > 0$, $\exists N$ s.t.

$$d(x_n, x_m) < \frac{\varepsilon}{2}, \quad \forall n, m \geq N.$$

As $n_j \to \infty$ as $j \to \infty$, $\exists\ j_1 \geq j_0$ s.t. $n_{j_1} \geq N$.

Hence $\forall n \geq N$,

$$d(x_n, z) \leq d(x_n, x_{n_{j_1}}) + d(x_{n_{j_1}}, z)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

$\therefore\ x_n \to z \in K$

$\therefore\ K$ is compact. ※

<u>eg3.1</u>: • $(\mathbb{R}$, standard$)$ is complete

• $[a,b]$, $(-\infty, b]$, $[a, \infty)$ complete

• $[a,b)$ not complete $\left(\because x_n = b - \frac{1}{n} \to b \notin [a,b)\right)$

• $\mathbb{Q}$ is not complete.

<u>eg3.2</u> $(X = C[a,b], d_\infty)$ is complete:

Cauchy seq $\{f_n\}$ in d_∞-metric

$\Leftrightarrow \forall \varepsilon > 0, \exists n_0$ s.t.

$$\max_{[a,b]} |f_n(x) - f_m(x)| < \varepsilon, \ \forall n, m \geq n_0$$

$\therefore\ f_n(x) \to f(x)$ uniformly for some $f \in C[a,b]$

※