Ch8 Morse index form and Bonnet–Myers Theorem

Let γ = normalized geodesic defined on $[a,b]$

$\mathcal{V} = \mathcal{V}(a,b) = \{ X = \text{piecewise } C^\infty \text{ vector field along } \gamma$
s.t. $\langle X, \gamma' \rangle = 0 \}$

$\mathcal{V}_0 = \mathcal{V}_0(a,b) = \{ X \in \mathcal{V} : X(a) = X(b) = 0 \}$
= the space of transversal vector fields of normal variations of γ.

Def: (1) $I(X,X) = I_a^b(X,X)$

$$= \int_a^b \left[|X'(t)|^2 - \langle R_{\gamma' X}\gamma', X \rangle \right] dt$$

$\forall X \in \mathcal{V}$.

(Note: $\int_a^b |X'(t)|^2$ means $\sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} |X'(t)|^2 dt$
where $a = a_0 < a_1 < \cdots < a_k = b$ s.t. $X\big|_{[a_i, a_{i+1}]} \in C^\infty$)

(2) $I(X,Y) \overset{\text{def}}{=} \frac{1}{2}\left[I(X+Y, X+Y) - I(X,X) - I(Y,Y) \right]$ $\quad \forall X, Y \in \mathcal{V}$

$\quad \parallel$

$I_a^b(X,Y)$

is called the index form of γ.

Notes: (i) $I(X,Y) = \int_a^b \left[\langle X', Y' \rangle - \langle R_{\gamma' X}\gamma', Y \rangle \right] dt$

(Ex!)

(ii) $I(X,Y)$ is bilinear (wrt scalar multiplication) and symmetric (Ex!)

(iii) If U = transversal vector field of a normal variation $\{\gamma_u\}$ of the normalized geodesic γ, then $U \in \mathcal{V}_0 \subset \mathcal{V}$ and the 2nd variation

$$L''(0) = I(U,U) \quad \left(\text{by } 2^{\text{nd}} \text{ variation formula}\right)$$

Lemma 1: Let
- $\gamma = [a,b] \to M$ normalized geodesic
- $\gamma(b)$ conjugate to $\gamma(a)$

Then $\forall$ normal Jacobi field U with $U(a) = U(b) = 0$ satisfies $I(U,U) = 0$.

Pf:
$$\begin{aligned}
I(U,U) &= \int_a^b |U'|^2 - \langle R_{\gamma' U}\gamma', U \rangle \\
&= \int_a^b |U'|^2 + \langle U'', U \rangle \quad \text{(Jacobi eqt.)} \\
&= \int_a^b |U'|^2 + \langle U', U \rangle' - |U'|^2 \\
&= \langle U', U \rangle \Big|_a^b = 0.
\end{aligned}$$
※

<u>Note</u>: Therefore, if $\gamma(b)$ conjugate to $\gamma(a)$, then the index form of γ is <u>degenerate</u>.

<u>Terminology</u>: A geodesic $\gamma : [a,b] \to M$ is said to <u>contain no conjugate point</u> if $\gamma(a)$ has no conjugate point along γ.

<u>Lemma 2</u>: Let
- $\gamma : [a,b] \to M$ normalized geodesic
- γ has no conjugate point

Then $I(X, Y)$ is <u>positive definite</u> on $\mathcal{V}_0(a,b)$.

<u>Lemma 3</u> Let
- $\gamma : [a,b] \to M$ normalized geodesic
- $\gamma(b)$ conjugate to $\gamma(a)$
- $\gamma(c)$ is not conjugate to $\gamma(a)$, $\forall c \in (a,b)$.

Then $I(X, Y)$ is <u>semi-positive definite</u> on $\mathcal{V}_0(a,b)$ but <u>not</u> positive definite.

<u>Lemma 4</u> Let
- $\gamma : [a,b] \to M$ normalized geodesic

Then $\exists\, c \in (a,b)$ s.t. $\gamma(c)$ is conjugate to $\gamma(a)$

$\iff \exists\, X \in \mathcal{V}_0(a,b)$ s.t. $I(X, X) < 0$.

Cor: If $\gamma : [a,b] \to M$ is a normalized geodesic which contains no conjugate point, then $\forall [\alpha,\beta] \subset [a,b]$, $\gamma|_{[\alpha,\beta]}$ has no conjugate point.

Pf: Suppose not, then $\exists [\alpha,\beta]$ s.t. $\gamma(\beta)$ conjugate to $\gamma(\alpha)$. Then by lemma 3, $\exists J \not\equiv 0 \in \mathcal{V}_0(\alpha,\beta)$ s.t. $I(J,J)=0$ $\quad (J(\alpha)=J(\beta)=0)$

Define a piecewise C^∞ vector field X along $\gamma : [a,b] \to M$ by

$$X = \begin{cases} J, & t \in [\alpha,\beta] \\ 0, & \text{otherwise} \end{cases}$$

Then X is well-defined and belongs to $\mathcal{V}_0(a,b)$

$$\begin{aligned} I_a^b(X,X) &= \int_a^b |X'|^2 - \langle R_{\gamma' X}\gamma', X\rangle \\ &= \int_\alpha^\beta |J'|^2 - \langle R_{\gamma' J}\gamma', J\rangle \\ &= I_\alpha^\beta(J,J) = 0. \end{aligned}$$

Hence Lemma 2 $\Rightarrow$ $\gamma : [a,b] \to M$ contains conjugate point.

Contradiction $\times\!\!\!\times$

To prove Lemmas 2–4, we need the following

<u>Claim</u>: For $X, Y \in C^\infty$

$$(*) \quad I_a^b(X,Y) = \langle X', Y\rangle\Big|_a^b - \int_a^b \langle X'' + R_{\gamma' X}\gamma', Y\rangle\, dt$$

<u>Pf</u>: $I(X,Y) = \int_a^b \langle X', Y'\rangle - \langle R_{\gamma' X}\gamma', Y\rangle$

$$= \int_a^b \langle X', Y\rangle' - \langle X'', Y\rangle - \langle R_{\gamma' X}\gamma', Y\rangle$$

$$= \langle X', Y\rangle\Big|_a^b - \int_a^b \langle X'' + R_{\gamma' X}\gamma', Y\rangle \quad \#$$

<u>Claim</u>: For piecewise C^∞ X Y, with

$X_i = X\big|_{[a_i, a_{i+1}]} \in C^\infty$ where $a = a_0 < a_1 < \cdots < a_k = b$,

$$(*) \quad I(X,Y) = \sum_{i=0}^{k-1} \langle X_i', Y\rangle\Big|_{a_i}^{a_{i+1}} - \sum_{i=0}^{k-1} \int_{a_i}^{a_{i+1}} \langle X_i'' + R_{\gamma' X_i}\gamma', Y\rangle\, dt$$

<u>Lemma 5</u>: Let
- $\gamma : [a,b] \to M$ normalized geodesic
- $U \in \mathcal{V}(a,b)$

Then $\quad I(U, \mathcal{V}_0) = 0 \iff U$ is a Jacobi field.

<u>Pf</u>: $(\Leftarrow)$ By $(*)$

$$I(V,Y)=\sum_{i=0}^{k-1}\langle V',Y\rangle\Big|_{a_i}^{a_{i+1}}-\sum_{i=0}^{k-1}\int_{a_i}^{a_{i+1}}\langle V''+R_{\gamma' V}\gamma',Y\rangle\,dt$$

(Jacobi field $V\in C^\infty$)
$Y(a)=Y(b)=0$

$$=0-0 \qquad \text{because } V''+R_{\gamma' V}\gamma'=0.$$

($\Rightarrow$) Suppose $I(V,\mathcal{V}_0)=0$

Since V is piecewise C^∞, $\exists\ a=a_0<a_1<\cdots<a_k=b$.

s.t. $V_i=V|_{[a_i,a_{i+1}]}\in C^\infty$, $i=0,\cdots,k-1$.

Take a C^∞ function f on $[a,b]$ s.t.

$$\begin{cases} f(a_i)=0, & \forall i=0,\cdots,k-1 \\ f>0 & \text{otherwise}\end{cases}$$

Let $X=V$, $Y=f(V''+R_{\gamma' V}\gamma')$

Then Y is well-defined & $\in\mathcal{V}_0$

Hence $(*)\Rightarrow$

$$0=I(V,Y)=\sum_{i=0}^{k-1}\langle V_i',Y\rangle\Big|_{a_i}^{a_{i+1}}-\sum_{i=1}^{k-1}\int_{a_i}^{a_{i+1}}\langle V''+R_{\gamma' V}\gamma',f(V''+R_{\gamma' V}\gamma')\rangle$$

$$=-\sum_{i=1}^{k-1}\int_{a_i}^{a_{i+1}}f\,|V''+R_{\gamma' V}\gamma'|^2 \qquad \left(\begin{array}{l}\text{since } Y(a_i)=0\\ \forall i.\end{array}\right)$$

$\Rightarrow\ V''+R_{\gamma' V}\gamma'=0$ on $[a_i,a_{i+1}]$, $\forall i=0,\cdots,k-1$.

Putting it back to the formula (*), one has

$$0 = I(V, \tilde{Y}) = \sum_{i=0}^{k-1} \langle V', \tilde{Y} \rangle \Big|_{a_i}^{a_{i+1}} , \qquad \forall \tilde{Y} \in \mathcal{V}_0$$

For a fixed $i_0 \in \{1, \cdots, k-1\}$, take $\tilde{Y}_{i_0} \in \mathcal{V}_0$

s.t.

$$\begin{cases} \tilde{Y}_{i_0}(a_i) = 0 , & \forall i \neq i_0 \\ \tilde{Y}_{i_0}(a_{i_0}) = V'_{i_0+1}(a_{i_0}) - V'_{i_0}(a_{i_0}) \end{cases}$$

Then

$$0 = I(V, \tilde{Y}) = -\langle V'_{i_0+1}(a_{i_0}), \tilde{Y}_{i_0}(a_{i_0}) \rangle + \langle V'_{i_0}(a_{i_0}), \tilde{Y}_{i_0}(a_{i_0}) \rangle$$

$$= -|\tilde{Y}_{i_0}(a_{i_0})|^2$$

$$\Rightarrow \quad V'_{i_0+1}(a_{i_0}) = V'_{i_0}(a_{i_0}).$$

Since $i_0 \in \{1, \cdots, k-1\}$ is arbitrary, V is in fact C^1.

Then existence & uniqueness thm of ODE $\Rightarrow$ V is Jacobi. #

Proof of Lemma 2

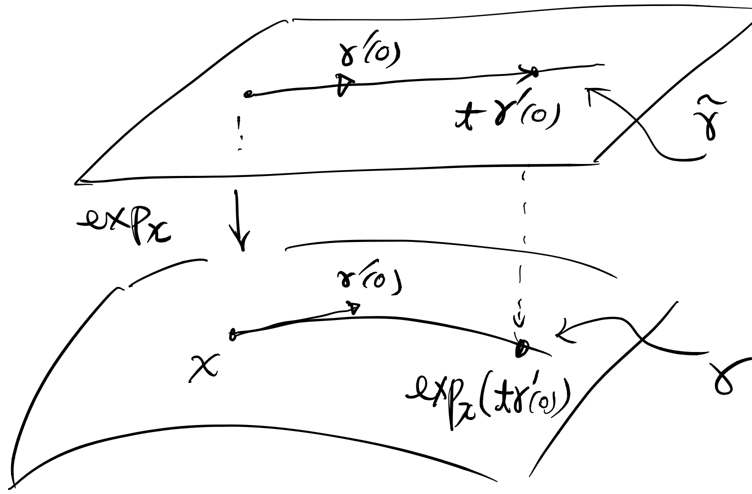
We may assume $a=0$, i.e. $\gamma : [0,b] \to M$.

Define

$$\begin{aligned} \tilde{\gamma} : [0,b] &\to T_xM \\ t &\mapsto t\gamma'(0) \end{aligned}$$

where $x = \gamma(0)$, $|\gamma'(0)| = 1$

By assumption, γ has no conjugate point, hence $d\exp_x$ has no singular point along $\tilde{\gamma}$.



$\Rightarrow \exists$ nbd. $\mathcal{U}$ of $\tilde{\gamma}([0,b])$ in T_xM s.t.

$$\exp_x : \mathcal{U} \to M \text{ is an immersion}$$

Then same proof as in Thm 2 of Ch4, one can show that

(**) For any piecewise C^∞ curve $\sigma : [0,b] \to \exp_x \mathcal{U}$ connecting x to $\gamma(b)$, $L(\sigma) \geq L(\gamma)$. And equality holds $\Leftrightarrow \sigma =$ monotonic reparametrization of γ. (Ex!)

Now for any $\overbrace{\text{variation}}^{\text{normal}}$ $\{\gamma_u\}$, $u \in (-\varepsilon, \varepsilon)$. With $\varepsilon > 0$ small enough, we may assume $\gamma_u \subset \exp_x \mathcal{U}$. Then by

(**) $\qquad L(u) \geq L(0)$, $\forall u \in (-\varepsilon, \varepsilon)$

Since $L(u)$ is C^∞, $\displaystyle L''(0) = \lim_{s \to 0} \frac{L(-s) + L(s) - 2L(0)}{s^2} \geq 0$

Noting that any $X \in \mathcal{V}_0$ is a transversal vector field of a normal variation of γ, therefore

$$I(X,X) = L''(0) \geq 0 , \qquad \forall X \in \mathcal{V}_0$$

Suppose that $I(X,X)=0$, we have $\forall \varepsilon > 0$, $Y \in \mathcal{V}_0$

$$0 \leq I(X \pm \varepsilon Y, X \pm \varepsilon Y) = I(X,X) \pm 2\varepsilon I(X,Y) + \varepsilon^2 I(Y,Y)$$

$$= \pm 2\varepsilon I(X,Y) + \varepsilon^2 I(Y,Y)$$

$$\Rightarrow \quad -\varepsilon I(Y,Y) \leq 2 I(X,Y) \leq \varepsilon I(Y,Y), \ \forall \varepsilon > 0, Y \in \mathcal{V}_0$$

Letting $\varepsilon \to 0$, we have $I(X,Y)=0$, $\forall Y \in \mathcal{V}_0$

Lemma 5 $\Rightarrow$ X = Jacobi.

But $X(0) = X(b) = 0$ and $\gamma(b)$ is not conjugate to $\gamma(0)$,

$$X \equiv 0 .$$

$\therefore$ I is positive definite. ※

<u>Lemma 6</u> (Cor to Lemma 2) (<u>Minimality of Jacobi field</u>)

Suppose
- $\gamma = [a,b] \to M$ normalized geodesic
- γ has no conjugate point.
- U = Jacobi field along γ.

Then $\forall X \in \mathcal{V}(a,b)$ with $X(a) = U(a)$ & $X(b) = U(b)$,

$$I(U,U) \leq I(X,X)$$

Equality holds $\Leftrightarrow$ $X = U$.

Pf: Note $U - X \in \mathcal{V}_0(a,b)$

Lemma 2 $\Rightarrow$ $\quad 0 \leq I(U-X, U-X)$

$$= I(U,U) - 2I(U,X) + I(X,X).$$

$$I(U,U) = \langle U', U\rangle\Big|_a^b - \int_a^b \langle \cancelto{0}{U'' + R_{\gamma' U}\gamma'}, U\rangle$$

$$= \langle U', U\rangle\Big|_a^b$$

$$I(U,X) = \langle U', X\rangle\Big|_a^b - \int_a^b \langle \cancelto{0}{U'' + R_{\gamma' U}\gamma'}, X\rangle$$

$$= \langle U', X\rangle\Big|_a^b = \langle U', U\rangle\Big|_a^b = I(U,U)$$

$\therefore \quad 0 \leq I(U,U) - 2I(U,U) + I(X,X)$

$\Rightarrow \quad I(U,U) \leq I(X,X).$

Equality $\Leftrightarrow$ $\quad 0 = I(U-X, U-X) \Leftrightarrow U = X.$ #

Proof of Lemma 3

It is clear that $I(X,Y)$ is not positive definite (by Lemma 1).

Take a parallel frame field $\{E_1(t), \cdots, E_n(t)\}$ along γ s.t. $E_1(t) = \gamma'(t)$.

Then $\forall X \in \mathcal{V}_0(0,b)$

$$X(t)=\sum_{i=2}^{n} f_i(t)E_i(t) \quad \text{with } f_i(0)=f_i(b)=0.$$

$\forall \beta\in[0,b]$, define $\tau(X)\in \mathcal{V}_0(0,\beta)$ by

$$\tau(X)(t)=\sum_{i=2}^{n} f_i\left(\frac{b}{\beta}t\right)E_i\left(\frac{b}{\beta}t\right)$$

Then

$$I_0^\beta(\tau(X),\tau(X))=\sum_{i=2}^{n}\int_0^\beta \left|\frac{d}{dt}f_i\left(\frac{b}{\beta}t\right)\right|^2$$

$$-\sum_{i,j} f_i\left(\frac{b}{\beta}t\right)f_j\left(\frac{b}{\beta}t\right)\left\langle R_{\gamma'(t)E_i(\frac{b}{\beta}t)}\gamma'(t), E_j\left(\frac{b}{\beta}t\right)\right\rangle$$

So $\lim_{\beta\to b} I_0^\beta(\tau(X),\tau(X))=I_0^b(X,X)$

Since $\gamma(b)$ is the only conjugate point, Lemma 2

$\Rightarrow I_0^\beta(\tau(X),\tau(X))\geq 0$.

Hence $I_0^b(X,X)\geq 0$, i.e. I_0^b is semi-positive definite ✻

To prove Lemma 4, we need

<u>Lemma 7</u> Let • $\gamma : [0,b] \to M$ normalized geodesic

• $\gamma(b)$ is not conjugate to $\gamma(0)$

Then $\forall\ v \in T_{\gamma(b)}M$, $\exists!$ Jacobi field V along γ s.t. $V(0)=0$ and $V(b)=v$.

(Pf: Ex!)

<u>Proof of Lemma 4</u>

($\Rightarrow$) If $\exists\ c \in (a,b)$ s.t. $\gamma(c)$ conjugate to $\gamma(a)$.

Then $\exists$ non-trivial normal Jacobi field J_1 along γ s.t. $J_1(a) = J_1(c) = 0$.

Define $J \in \mathcal{V}_0(a,b)$ by

$$J = \begin{cases} J_1, & t \in [a,c] \\ 0, & t \in [c,b] \end{cases}$$

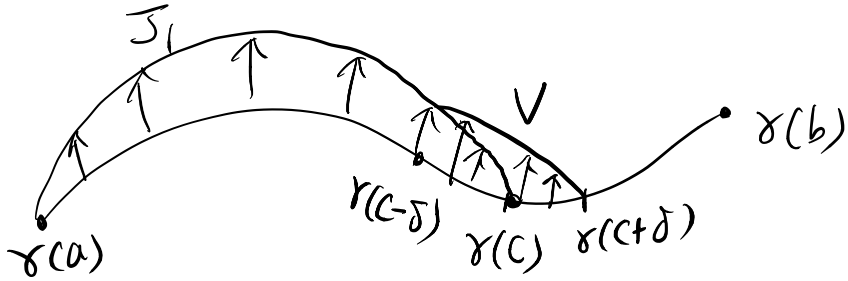
Then $I_a^b(J,J) = I_a^c(J_1,J_1) + I_c^b(0,0) = 0$

Now take $\delta > 0$ small s.t.

$$\exp_{\gamma(c+\delta)} : T_{\gamma(c+\delta)}M \to M$$

is diffeo. on $B(3\delta) \subset T_{\gamma(c+\delta)}M$ (& $c+\delta < b$)

Since $d(\gamma(c-\delta), \gamma(c+\delta)) \leq 2\delta$, $\gamma(c-\delta)$ is not conjugate to $\gamma(c+\delta)$.



Then Lemma 7 $\Rightarrow$ $\exists!$ Jacobi field V s.t.

$$V(c+\delta) = 0 \ \ \& \ \ V(c-\delta) = J(c-\delta) = J_1(c-\delta)$$

Define $U = \begin{cases} J_1, & t \in [a, c-\delta] \\ V, & t \in [c-\delta, c+\delta] \\ 0, & t \in [c+\delta, b] \end{cases}$

Then $I_a^b(U,U) = I_a^{c-\delta}(J_1,J_1) + I_{c-\delta}^{c+\delta}(V,V) + I_{c+\delta}^b(0,0)$

$$\wedge$$

$$\left(I_{c-\delta}^{c+\delta}(J,J) \text{ by lemma 6} \right)$$

$$< I_a^{c-\delta}(J,J) + I_{c-\delta}^{c+\delta}(J,J) + I_{c+\delta}^b(J,J)$$

$$= I_a^b(J,J) = 0. \ \ ※$$

($\Leftarrow$) If $\exists\, U \in \mathfrak{V}_0(a,b)$ s.t. $I(U,U) < 0$ then Lemmas 2 & 3 $\Rightarrow$ $\exists$ conjugate point to $\gamma(a)$ in $\gamma((a,b))$ ※

Fact (Ex!) Applying Lemma 4 to S^2, show that if $b > \pi$, then $\exists$ a piecewise smooth $f_0 : [0,b] \to \mathbb{R}$ s.t.

$$(***) \qquad \begin{cases} f_0(0) = f_0(b) = 0 \\ \int_a^b \left[(f_0')^2 - f_0^2 \right] < 0 \end{cases}$$

Thm 8 (Bonnet - Myers)

Let
- M = complete Riem. mfd.
- $\mathrm{Ricci}_M \geq (n-1)c$, $c > 0$

Then M is compact and $\mathrm{diam}(M) \leq \frac{\pi}{\sqrt{c}}$.

Pf: Scaling $\Rightarrow$ we may assume $c = 1$.

Then we need to show that if $\gamma : [0,b] \to M$ normalized shortest geodesic connecting x to y, then $b \leq \pi$.

Take parallel frame field $\{E_1(t), \cdots, E_n(t)\}$ along γ

s.t. $E_1(t) = \gamma'(t)$.

If $b > \pi$, define

$$X_i(t) = f_0(t) E_i(t), \quad i = 2, \cdots, n.$$

where $f_0(t)$ is the function in (***).

Then $X_i \in \mathcal{V}_0(0,b)$ $\forall\, i = 2, \cdots, n$ and

$$\sum_{i=2}^{n} I(X_i, X_i) = \sum_{i=2}^{n} \int_0^b \left(|X_i'|^2 - \langle R_{\gamma' X_i} \gamma', X_i \rangle \right) dt$$

$$= (n-1) \int_0^b (f_0')^2 - f_0^2 \int_0^b \sum_{i=2}^{n} \langle R_{E_1 E_i} E_1, E_i \rangle dt$$

$$\leq (n-1) \int_0^b \left((f_0')^2 - f_0^2 \right) < 0$$

$\uparrow$

($\mathrm{Ricci}_M \geq n-1$)

$\Rightarrow$ $\exists\, i_0$ s.t. $I(X_{i_0}, X_{i_0}) < 0$.

$\Rightarrow$ γ is not minimizing.

Contradiction.

$\therefore$ $b \leq \pi$ $\#$