$$D_{\gamma'(t)} X = 0 \iff \frac{dX^k}{dt} + \left(\Gamma^k_{ij} \gamma'^i\right) X^j = 0, \quad \forall k = 1, \dots, n$$

in local coordinates.

which is a linear ODE system in $X^1, \dots, X^n$.

<u>Linear ODE theory</u> $\Rightarrow$

$\forall\, v \in T_{\gamma(t)} M$, $\exists!$ soln $X(t)$ to the IVP

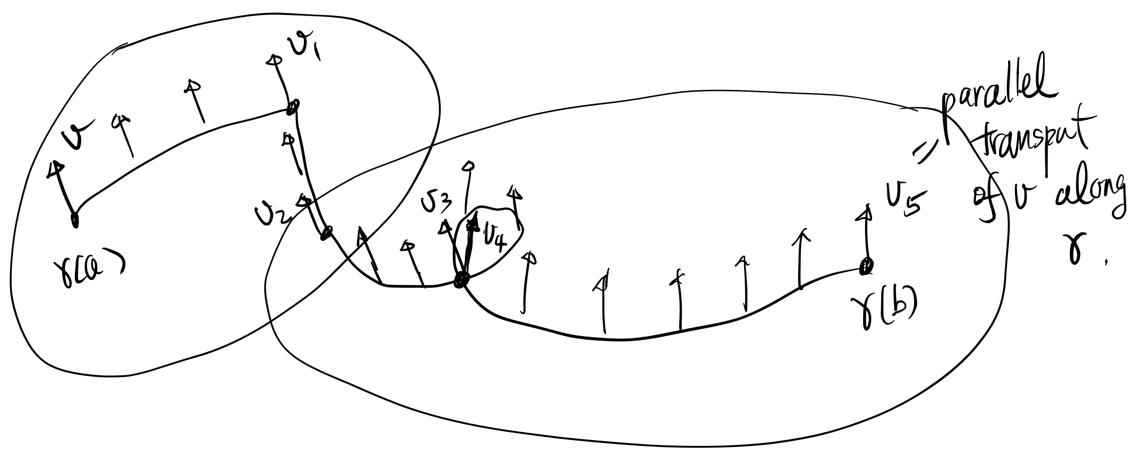
$$\begin{cases} D_{\gamma'(t)} X = 0, & \forall t \in [a, b]. \\ X(a) = v \end{cases}$$

<u>Def</u>: A vector field X along γ is called <u>parallel</u> if $D_{\gamma'} X = 0$.

<u>Def</u>: A vector $w \in T_{\gamma(b)} M$ is called a <u>parallel transport</u> of a vector $v \in T_{\gamma(a)} M$ <u>along γ</u> if $\exists$ a parallel vector field X along γ such that $X(a) = v$ & $X(b) = w$.

<u>Notes</u>: (i) parallel transport w of v (along γ) is <u>uniquely</u> determined by v (for fixed γ).

(ii) If γ is <u>not</u> embedded, or <u>not</u> contained in a chart, or γ is only piecewise smooth, we can use subdivision to define parallel transport of a vector $v \in T_{\gamma(a)}M$ along γ.



(v_3 may not equal to v_4 after a loop!)

Hence

Thm $\forall$ immersed curve $\gamma : [a,b] \to M$ & $v \in T_{\gamma(a)}M$, $\exists!$ parallel vector field $X(t)$ along γ such that $X(a) = v$.

Hence, $\exists!$ $w \in T_{\gamma(b)}M$ such that w is the parallel transport of v along γ.

This Thm $\Rightarrow$ one can define, $\forall$ immersed curve $\gamma : [a,b] \to M$, a mapping

$$\begin{array}{ccc} P^\gamma : T_{\gamma(a)}M & \longrightarrow & T_{\gamma(b)}M \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ v & \longmapsto & w = \text{parallel transport of} \\ & & v \text{ along } \gamma. \end{array}$$

Thm : $P^\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is an vector space isomorphism.

(Pf : Ex)

- P^γ is called parallel transport from $\gamma(a)$ to $\gamma(b)$ along γ.

- Furthermore, if D is the Levi-Civita connection of a metric g on M, then for any 2 vector fields X & Y along γ (γ embedded),

$$\begin{aligned} \frac{d}{dt}\langle X, Y\rangle &= \gamma'(t)\langle X, Y\rangle \\ &= \langle D_{\gamma'(t)}X, Y\rangle + \langle X, D_{\gamma'(t)}Y\rangle \\ &= 0 \quad \text{if both } X \,\&\, Y \text{ are parallel.} \end{aligned}$$

$\therefore$ $P^\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M$ is in fact an isometry of the inner product spaces.

Conversely, if D is a connection such that all P^γ are isometries of the inner product spaces, then $\forall$ vector fields X, Y, Z, we choose a curve $\gamma: [0,1] \to M$ such that $\gamma'(0) = X(x)$ $(x \in M)$.

Let $\{e_1, \cdots, e_n\}$ be an orthonormal basis of T_xM. Then parallel transport P^γ along γ defines orthonormal basis $\{e_1(t), \cdots, e_n(t)\}$ of $T_{\gamma(t)}M$, $\forall\, t \in [0,1]$ (P^γ are isometries $\forall\, t$). Hence

$$Y(\gamma(t)) = \sum Y^i(t)\, e_i(t)$$

$$Z(\gamma(t)) = \sum Z^i(t)\, e_i(t)$$

for some $Y^i(t)$ & $Z^i(t)$.

$$\Rightarrow \quad X(x)\langle Y, Z\rangle = \gamma'(0)\langle Y, Z\rangle$$

$$= \frac{d}{dt}\Big|_{t=0} \langle Y, Z\rangle(\gamma(t))$$

$$= \frac{d}{dt}\Big|_{t=0} Y^i(t) Z^i(t)$$

$$= \frac{dY^i}{dt}(0)\, Z^i(0) + Y^i(0)\, \frac{dZ^i}{dt}(0).$$

Note that $D_{\gamma'(0)} Y = D_{\gamma'(0)}\left(\sum Y^i(t)\, e_i(t)\right)$

$$= \sum \left[\frac{dY^i}{dt}(0)\, e_i(0) + Y^i(0) \cancelto{0}{D_{\gamma'(0)} e_i} \right]$$

$$= \sum \frac{dY^i}{dt}(0)\, e_i$$

Similarly for $D_{\gamma'(0)} Z$.

Hence $\quad X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$.

i.e. D is compatible with the metric g.

<u>Conclusion</u>: D = compatible with g

$\iff$ P^γ = isometry, $\forall \gamma$.

In particular, if D is symmetric, then

$$D = \text{Levi-Civita} \iff P^\gamma = \text{isometry}, \forall \gamma.$$

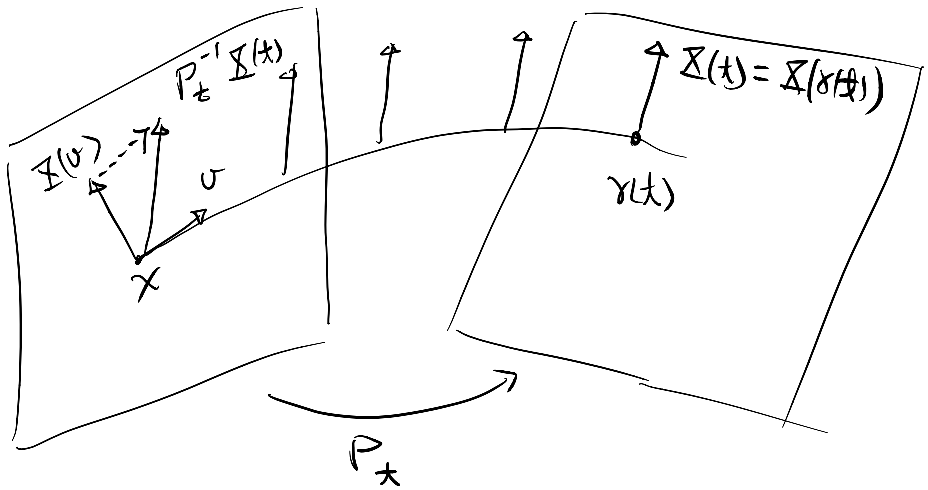
<u>Thm</u>: $\forall\, v \in T_xM$ & $X \in \Gamma(TM)$ $\qquad$ (for D = Levi-Civita)

$$D_v X = \left.\frac{d}{dt}\right|_{t=0} P_t^{-1} X(\gamma(t))$$

where $\gamma : [0,1] \to M$ is a curve such that

$\gamma(0) = x$, $\gamma'(0) = v$

$P_t : T_xM \to T_{\gamma(t)}M$ = parallel transport along $\gamma|_{[0,t]}$.



Pf: Let $\{e_i\}$ be an orthonormal basis of T_xM

Define $e_i(t) = P_t e_i$.

Then $\{e_i(t)\}$ is an o.n. basis of $T_{\gamma(t)}M$

Write X in terms of $\{e_i(t)\}$:

$$X(\gamma(t)) = \sum X^i(t)\, e_i(t) \quad \text{for some } X^i(t).$$

$$\Rightarrow \quad D_v X = \sum \frac{dX^i}{dt}(0)\, e_i$$

And
$$P_t^{-1}(X(\gamma(t))) = \sum X^i(t)\, P_t^{-1}(e_i(t))$$
$$= \sum X^i(t)\, e_i$$

$$\Rightarrow \quad \left.\frac{d}{dt}\right|_{t=0} P_t^{-1}(X(\gamma(t))) = \sum \frac{dX^i}{dt}(0)\, e_i = D_v X$$

$\#$

§2.3 Geodesic

Def: A curve $\gamma : [a,b] \to M$ is called a geodesic wrt the connection D if $\gamma'(t)$ is parallel along γ.

In local coordinates $\{x^i\}$

$$\gamma(t) = (x^1(t), \cdots, x^n(t))$$

$$\Rightarrow \quad \gamma'(t) = \sum \frac{dx^i}{dt}(t) \cdot \frac{\partial}{\partial x^i}\Big|_{\gamma(t)}$$

Hence

$$D_{\gamma'(t)} \gamma'(t) = \sum_k \left[\frac{d}{dt}\left(\frac{dx^k}{dt}\right) + \Gamma^k_{ij}(\gamma(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} \right] \frac{\partial}{\partial x^k}$$

$\therefore$ γ is a geodesic (wrt D)

$\Leftrightarrow D_{\gamma'} \gamma' = 0$

$$\Leftrightarrow \quad \frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}(x^1, \cdots, x^n) \frac{dx^i}{dt} \frac{dx^j}{dt} = 0, \quad \forall k = 1, \cdots, n.$$

which is a non-linear ODE system for $(x^1(t), \cdots, x^n(t))$.

ODE theory $\Rightarrow$

Lemma: $\forall$ connection D on M,

$\forall\ v \in T_x M$

$\Rightarrow \exists!$ geodesic $\gamma(t)$ wrt D on some interval $(-\varepsilon, \varepsilon)$ such that

$$\begin{cases} \gamma(0) = x \\ \gamma'(0) = v. \end{cases}$$

Note: If D is Levi-Civita connection of g. Then

$\forall$ geodesic γ of D, we have

$$\frac{d}{dt}\langle \gamma', \gamma' \rangle = \langle D_{\gamma'} \gamma', \gamma' \rangle + \langle \gamma', D_{\gamma'} \gamma' \rangle = 0$$

$\Rightarrow \quad |\gamma'(t)|$ is constant.

§2.4 Induced connection

Let M = Riemannian manifold

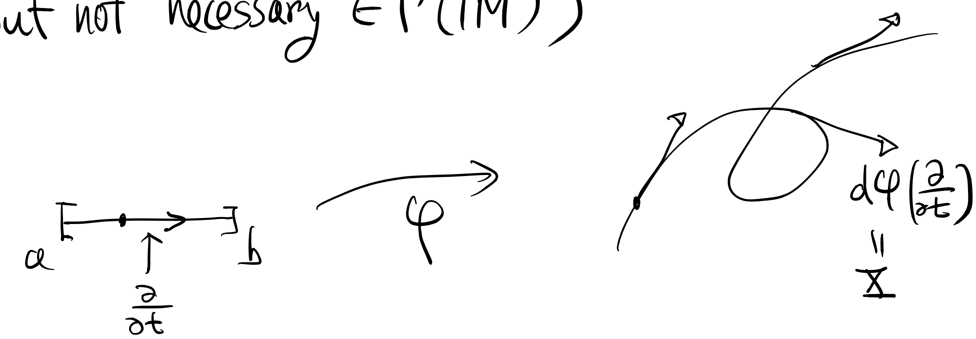
N = differentiable manifold

and $\varphi: N \to M$ C^∞ map.

Def: A map $X: N \to TM$ is called a vector field along φ if $\forall\ x \in N$, $X(x) \in T_{\varphi(x)} M$

$$\begin{tikzcd} & TM \arrow[d, "\pi"] \\ N \arrow[ur, "X"] \arrow[r, "\varphi"] & M \end{tikzcd}$$

eg : $\forall Y \in \Gamma(TN)$, $X = d\varphi(Y)$ is a vector field along φ
(but not necessary $\in \Gamma(TM)$)



Note: If $v \in T_x N$, and $\{E_i\}_{i=1}^n$ is a "frame field"
in a nbd V of $\varphi(x) \in M$.

(i.e. $\{E_i(p)\}$ is a basis of T_pM, $\forall p \in V$)
and $E_i(p)$ are smooth in p.

Then $\forall\ x \in \varphi^{-1}(V) \subset N$,

$$X(x) = \sum X^i(x) E_i(\varphi(x)) \in T_{\varphi(x)}M$$ for some functions $X^i(x)$ on N

Define

$$\tilde{D}_v X = \sum \left[v(X^i)(x) E_i(\varphi(x)) + X^i(x)\, D_{d\varphi(v)} E_i \right]$$

where D = Levi-Civita connection on M.

Fact: $\tilde{D}_v X$ is well-defined (indep. of the choice of

$\{E_i\}$)   (Pf: Ex.)

Def: • $\tilde{D}$ is called the induced connection.

• $\forall\ V \in \Gamma(TN)$, X = vector field along φ,

$$(\tilde{D}_V X)(x) \stackrel{\text{def}}{=\!=} \tilde{D}_{V(x)} X .$$

Fact: Since D = Levi-Civita on M,

• $\forall\ X, Y \in \Gamma(TN)$

$$\tilde{D}_X\, d\varphi(Y) - \tilde{D}_Y\, d\varphi(X) - d\varphi([X,Y]) = 0$$

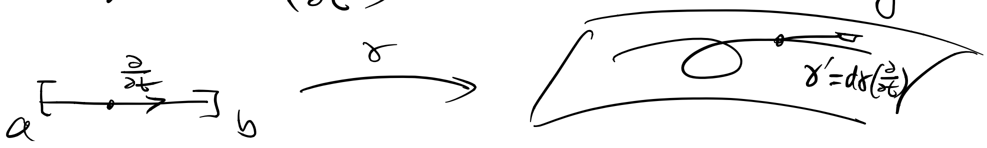
$$\Big(d\varphi([X,Y]) = [d\varphi(X), d\varphi(Y)] \Big)$$

• $\forall\ V, W$ vector fields along φ and $u \in T_xN$,

$$u\langle V, W\rangle = \langle \tilde{D}_u V, W\rangle + \langle V, \tilde{D}_u W\rangle .$$

Note: If $\gamma : [0,1] \to M$ is a smooth curve (not necessary embedded) then

$$\gamma' = d\gamma\left(\frac{\partial}{\partial t}\right) \text{ is a vector field along } \gamma$$



$$\begin{tikzcd} & TM \arrow[d] \\ {[a,b]} \arrow[ur, "d\gamma(\frac{\partial}{\partial t})"] \arrow[r, "\gamma"'] & M \end{tikzcd}$$

We define $D_{\gamma'}\gamma' \overset{\text{def}}{=\!=} \tilde{D}_{\frac{\partial}{\partial t}}\gamma'$

(check: If γ is embedded, this definition coincides with the previous one.)

$\therefore$ geodesic (and P^{γ}) can be defined for any smooth curve.

Ch3 Covariant derivative, Curvature Tensor

§3.1 Covariant derivative of tensors

Fact: let $\varphi: V \to W$ be an isomorphism between vector spaces, then φ can be extended to an isomorphism between the tensor algebras:

$$\tilde{\varphi}: \bigoplus_{r,s} T^{r,s}V \longrightarrow \bigoplus_{r,s} T^{r,s}W,$$

where $T^{r,s}V = (\underbrace{V\otimes\cdots\otimes V}_{r})\otimes(\underbrace{V^*\otimes\cdots\otimes V^*}_{s})$ $\quad V^* = \text{dual of } V.$

In fact, we can first define

$$\begin{array}{rccc} \varphi^*: & W^* & \longrightarrow & V^* \\ & \psi & & \psi \\ & \alpha & \longmapsto & \varphi^*(\alpha) \end{array} \quad \text{by} \quad \boxed{\varphi^*(\alpha)(v) = \alpha(\varphi(v))}$$

Then $\varphi = \text{isom} \Rightarrow \varphi^*$ isom

$\Rightarrow \quad (\varphi^*)^{-1}: V^* \to W^*$ exists.

Hence we can define

$\forall \; v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s \in T^{r,s}V,$

$$\tilde{\varphi}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)$$
$$= \varphi(v_1) \otimes \cdots \otimes \varphi(v_r) \otimes (\varphi^*)^{-1}(\alpha^1) \otimes \cdots \otimes (\varphi^*)^{-1}(\alpha^s) \in T^{r,s}W.$$

Finally, extend $\tilde{\varphi}$ to all $\bigoplus_{r,s} T^{r,s}V$ by linearity and can be checked that $\tilde{\varphi}$ is an isomorphism.

<u>Def</u>: Let M = Riemannian manifold, $x \in M$, $v \in T_xM$, γ = curve with $\gamma(0) = x$, $\gamma'(0) = v$.

Then $\forall$ tensor field K on M, we define the <u>covariant derivative</u> of K <u>wrt v</u> by

$$D_v K = \frac{d}{dt}\Big|_{t=0} (\widetilde{P^\gamma_t})^{-1}(K(\gamma(t)))$$

where $\widetilde{P}^{\gamma}_t : \bigoplus_{r,s} T^{r,s}(T_xM) \to \bigoplus_{r,s} T^{r,s}(T_{\gamma(t)}M)$

is the extension of the parallel transport

$P^{\gamma}_t : T_xM \to T_{\gamma(t)}M$ wrt Levi-Civita connection.

Caution: We need to check D_vK doesn't depend on γ.

Properties:

(1) If K is a (r,s)-tensor, then D_vK is also a (r,s)-tensor.

(2) D_v is a derivation on the tensor algebra:

$$D_v(K_1 \otimes K_2) = (D_vK_1) \otimes K_2 + K_1 \otimes (D_vK_2).$$

(3) D_v commutes with "contractions".

Def (of contraction) The contractions C_{pq}, $p=1,\cdots,r$, $q=1,\cdots,s$

are linear maps

$$C_{pq} : (\otimes^r TM) \otimes (\otimes^s T^*M) \to (\otimes^{r-1} TM) \otimes (\otimes^{s-1} T^*M)$$

defined by

$$C_{pq}(v_1 \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \alpha^s)$$

$$= \alpha^q(v_p)\, v_1 \otimes \cdots \otimes \hat{v}_p \otimes \cdots \otimes v_r \otimes \alpha^1 \otimes \cdots \otimes \hat{\alpha}^q \otimes \cdots \otimes \alpha^s$$

omitted

<u>egs</u>: For $C_{11}: TM \otimes T^*M \longrightarrow \mathbb{R} \ (\cong (\otimes^0 TM) \otimes (\otimes^0 T^*M))$

takes $C_{11}\left(\frac{\partial}{\partial x^i} \otimes dx^j\right) = dx^j\left(\frac{\partial}{\partial x^i}\right) = \delta_i^j \in \mathbb{R}$.

For $C_{11}: (TM) \otimes (\otimes^2 T^*M) \to T^*M$

$$\frac{\partial}{\partial x^i} \otimes (dx^{j_1} \otimes dx^{j_2}) \mapsto dx^{j_1}\left(\frac{\partial}{\partial x^i}\right) dx^{j_2} = \delta_i^{j_1} dx^{j_2}$$

Property (3) means if $\mathcal{C} = C_{pq}$ is a contraction, then

$$\boxed{D_v(\mathcal{C}K) = \mathcal{C}(D_v K)}$$

<u>Pf</u>: (1) is clear

(2) We do a special case only. The general case can be proved similarly.

Suppose $K = X \otimes Y \otimes \rho \in (\otimes^2 TM) \otimes (T^*M)$

i.e. X, Y are vector fields, ρ is a 1-form (linear combinations of dx^i)

Then we need to prove that

$$D_v K = (D_v X) \otimes Y \otimes \rho + X \otimes (D_v Y) \otimes \rho + X \otimes Y \otimes (D_v \rho).$$

Let $\{e_1(t), \cdots, e_n(t)\}$ be <u>parallel</u> vector field along γ s.t.

$\{e_i(t)\}$ forms a basis of $T_{\gamma(t)}M$,

Then $\forall t$, $\exists$ dual basis $\{\alpha^1(t), \cdots, \alpha^n(t)\}$ of $T^*_{\gamma(t)}M$,

i.e. $\alpha^i(t)(e_j(t)) = \delta^i_j$, $\forall t$.

<u>Claim</u>: $\{\alpha^i(t)\}$ are all parallel.

In fact, by definition of $\widetilde{P}_t$, we see that

$$\widetilde{P}_t(\alpha^i(0)) \overset{\text{def}}{=\!=} (P_t^*)^{-1}(\alpha^i(0))$$

$$\Leftrightarrow \quad P_t^*(\widetilde{P}_t(\alpha^i(0))) = \alpha^i(0)$$

$$\Leftrightarrow \quad P_t^*(\widetilde{P}_t(\alpha^i(0)))(e_j(0)) = \alpha^i(0)(e_j(0)) = \delta^i_j, \forall j$$

$$\Leftrightarrow \quad \widetilde{P}_t(\alpha^i(0))(P_t(e_j(0))) = \delta^i_j, \quad \forall j$$

$$\Leftrightarrow \quad \widetilde{P}_t(\alpha^i(0))(e_j(t)) = \delta^i_j, \quad \forall j$$

$$\Leftrightarrow \quad \widetilde{P}_t(\alpha^i(0)) = \alpha^i(t). \quad \#$$

Now, write

$$X(t) = X(\gamma(t)) = \sum X^i(t) e_i(t)$$
$$Y(t) = Y(\gamma(t)) = \sum Y^j(t) e_j(t)$$

$$\rho(t) = \rho(\gamma(t)) = \sum \rho_\ell(t)\,\alpha^\ell(t)$$

Then $K(t) = \sum_{i,j,\ell} X^i(t)\,Y^j(t)\,\rho_\ell(t)\; e_i(t)\otimes e_j(t)\otimes \alpha^\ell(t)$

$$\Rightarrow \left(\widetilde{P_t}\right)^{-1} K(t) = \sum_{i,j,\ell} X^i(t)\,Y^j(t)\,\rho_\ell(t)\; e_i(0)\otimes e_j(0)\otimes \alpha^\ell(0)$$

$$\Rightarrow D_v K = \frac{d}{dt}\Big|_{t=0} \left(\widetilde{P_t}\right)^{-1} K(t)$$

$$= \sum_{i,j,\ell}\left(\frac{dX^i}{dt}Y^j\rho_\ell + X^i\frac{dY^j}{dt}\rho_\ell + X^i Y^j\frac{d\rho_\ell}{dt}\right) e_i(0)\otimes e_j(0)\otimes\alpha^\ell(0)$$

Compare with
$$\begin{cases} D_v X = \sum \dfrac{dX^i}{dt} e_i(0) \\ D_v Y = \sum \dfrac{dY^j}{dt} e_j(0) \\ D_v \rho = \sum \dfrac{d\rho_\ell}{dt}\alpha^\ell(0), \end{cases}$$

we have $D_v K = (D_v X)\otimes Y\otimes\rho + X\otimes(D_v Y)\otimes\rho + X\otimes Y\otimes(D_v\rho)$

#