Notes: • In def 1, a tangent vector is represented by a curve γ. We usually write $\gamma'(0)$ for the tangent vector $[\gamma]$ for simplicity. (independent of chart!)

• In def 2, the "same" tangent vector will be represented in a chart (U, ϕ) by a vector $u \in \mathbb{R}^n$.

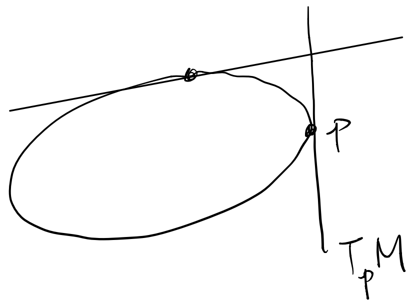
• Def 1 $\Leftrightarrow$ Def 2 by taking $u = (\phi \circ \gamma)'(0)$.

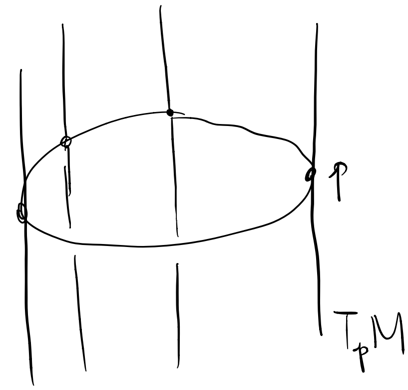
Notation: The set of tangent vectors to M at p is denoted by T_pM (Tangent space to M at $p \in M$)

Note: If a chart (U, ϕ) is given, then we have an "isomorphism"

$$\begin{aligned} \Theta_{U,\phi,p} : \mathbb{R}^n &\longrightarrow T_pM \\ u &\longmapsto [(U, \phi, u)]. \end{aligned}$$

Def: The disjoint union TM of T_pM, $\forall p \in M$, is called the tangent bundle of M.





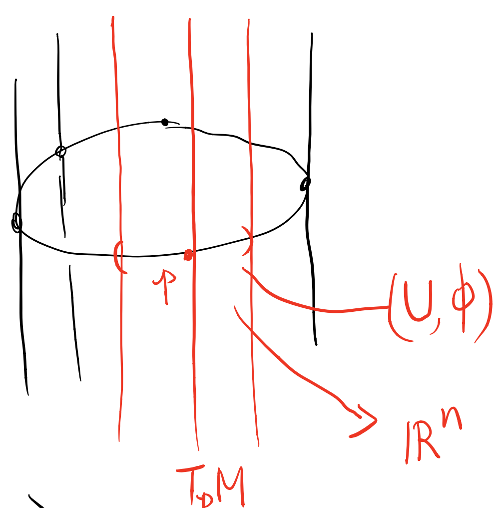
<u>Thm</u>: Let M be an n-dim'l C^k manifold ($k \geq 1$). Then TM can be equipped with a <u>$2n$</u>-dim'l C^{k-1} abstract manifold structure.

<u>Pf</u>: (Sketch)

For each chart (U, ϕ) of M,

define a "chart"

$$\left(\coprod_{p \in U} T_pM, \Phi \right) \text{ for } TM$$



by

$$\Phi(\xi_p) = \left(\phi(p), \theta^{-1}_{U,\phi,p}(\xi_p) \right) \in \mathbb{R}^n \times \mathbb{R}^n$$

$\forall\ \xi_p \in T_pM,\ p \in U$.

Then one can see all these "$\coprod_{p\in U} T_pM$" give a topology on TM such that Φ are homeomorphisms.

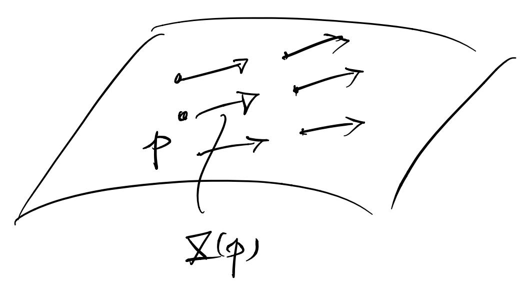
And one can check that TM is Hausdroff and $\left\{ \left(\coprod_{p\in U} T_pM, \Phi \right) \right\}_{(U,\phi)}$ forms an C^{k-1} altas of TM. (We've differentiated once in the equi. relation for tangent vectors.) $\#$

<u>Def</u>: A (<u>smooth</u>) <u>vector field</u> X on a manifold M is a smooth <u>section</u> of the tangent bundle TM of M, ie

$$X : M \longrightarrow TM$$

is a smooth map such that $X(p) \in T_pM$.

- The set of (smooth) vector fields on M is denoted by $\Gamma(TM)$.

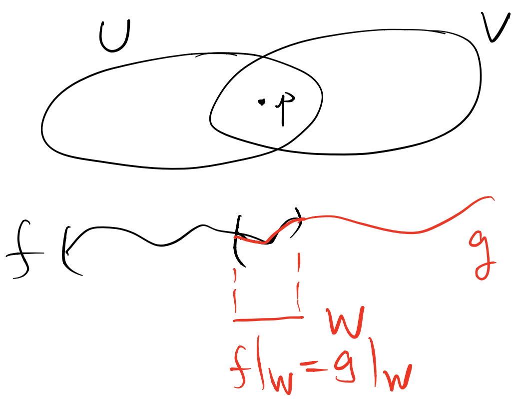


1.5 Tangent vectors as derivations

Let M be a smooth manifold, $p \in M$, consider C^∞ functions defined in a nbd. of p. Then we can define an equivalence relation:

$$f: U \to \mathbb{R} \sim g: V \longrightarrow \mathbb{R} \quad (p \in U \cap V)$$

$$\Longleftrightarrow \exists \text{ nbd } W \subset U \cap V \text{ of } p \text{ s.t. } f|_W = g|_W$$



<u>Def</u>: The equivalence classes for this relation are the <u>germs</u> of C^∞ functions at p. The space of germs of C^∞ functions at p is denoted by $\mathcal{C}_p^\infty(M)$.

Similarly, we can define $\mathcal{C}_p^0(M)$, $\mathcal{C}_p^k(M)$ & $\mathcal{C}_p^\omega(M)$ germs of <u>continuous</u>, <u>C^k</u>, and (real) <u>analytic functions</u>

respectively at p.

Remarks : • Space of functions has linear structure (and a product structure)

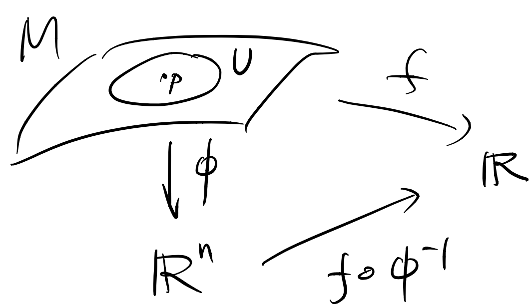
$\Rightarrow$ corresponding space of germs is a vector space (with a product structure).

• If M is a C^k manifold $(0 \leq k \leq \infty)$

then $\mathcal{E}^k_p(M) \cong \mathcal{E}^k_0(\mathbb{R}^n)$ (vector space isomorphism)

Pf: (Sketch)

germ of $f \longleftrightarrow$ germ of $f \circ \phi^{-1}$ for a chart (U, ϕ)



Def: A derivation on $\mathcal{E}^k_p(M)$ is a linear map

$\delta : \mathcal{E}^k_p(M) \to \mathbb{R}$ such that $\forall f, g \in \mathcal{E}^k_p(M)$

$\delta(fg) = f(p)\,\delta(g) + g(p)\,\delta(f)$.

where fg = product of the germs f & g

(Ex: How to define fg?)

<u>Notation</u>: We denote the set of derivations on $\mathcal{C}^k_p(M)$ by $\mathcal{D}^k_p(M)$, or $\mathcal{D}_p(M)$ if k is clear.

<u>Thm</u>: Any derivation of $\mathcal{C}^\infty_0(\mathbb{R}^n)$ can be written as

$$\delta(f) = \sum_{j=1}^{n} \delta(x^j) \frac{\partial f}{\partial x^j}(0)$$

germ

is a function representing the germ f.

Hence $\dim\left(\mathcal{D}^\infty_0(\mathbb{R}^n)\right) = n$

(where x^j = germ of the coordinate function
$$x^j : \mathbb{R}^n \to \mathbb{R}$$
$$\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \mapsto x^j \quad)$$

<u>Pf</u>: $\forall$ germ $f \in \mathcal{C}^\infty_0(\mathbb{R})$, f is represented by a C^∞ function, denoted by f again, in a nbd. of 0.

Then $f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx)\, dt$

$$= \int_0^1 \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(tx)\, x^j \, dt$$

$$= \sum_{j=1}^{n} x^j h_j(x) .$$

where $h_j(x) = \int_0^1 \frac{\partial f}{\partial x^j}(tx)\,dt$ is C^∞.

Then $\delta(f) = \delta(f - f(0))$ since $\delta(\text{const.}) = 0$ (Ex!)

$$= \delta\left(\sum_{j=1}^{n} x^j h_j(x)\right)$$

$$= \sum_{j=1}^{n} \cancelto{0}{x^j(0)}\,\delta(h_j) + h_j(0)\,\delta(x^j)$$

$$= \sum_{j=1}^{n} \delta(x^j)\, \frac{\partial f}{\partial x^j}(0) \quad \#$$

<u>Lemma</u> $\forall\, \xi \in T_pM$, $L_\xi(f) \overset{\text{def}}{=\!=} (D_p f)(\xi)$, $\forall f \in C_p^\infty(M)$.

Then $L_\xi \in \mathfrak{D}_p(M)$.

(Pf: Ex!) (Where $D_p f$ is the differential of a representation of f defined similarly as in Diff. Geom using def 1 of vector.)

$\underline{\text{Thm}}$: $T_pM \longrightarrow \mathcal{D}_p(M)$ is an isomorphism

$\quad\quad\ \ \xi \longmapsto L_\xi$ (between vector spaces)

$\underline{\text{Pf}}$: • $\xi \mapsto L_\xi$ is clear linear.

• $\ker(\xi \mapsto L_\xi) = 0$

$\underline{\text{Pf}}$: Let (U, ϕ) be a chart for M around p with $\phi(p) = 0 \in \mathbb{R}^n$. Then ξ can be represented by $\xi = (U, \phi, u)$ with $u \in T_0\mathbb{R}^n \cong \mathbb{R}^n$.

$\Rightarrow$ $\forall$ C^∞ function f in a nbd around p

$$L_\xi f = D_0(f \circ \phi^{-1})(u) \qquad (\text{Ex!})$$

$$= \sum_{j=1}^{n} u^j \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(0)$$

(where $u = (u^1, \cdots, u^n)$)

If $\xi \in \ker(\xi \mapsto L_\xi)$, then $\forall f$

$$0 = \sum_{j=1}^{n} u^j \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(0)$$

$\Rightarrow$ $u^j = 0$, $\forall j$ $\Rightarrow$ $\xi = 0$ ※

- Finally $\mathrm{Im}(\xi \mapsto L_\xi) = \mathcal{D}_p(M)$.

Pf: $\forall \delta \in \mathcal{D}_p(M) \cong \mathcal{D}_0(\mathbb{R}^n)$, by previous Thm

$$\Rightarrow \delta(f) = \sum_{j=1}^{n} \delta(x^j) \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(0)$$

$$\therefore \delta = L_\xi \text{ for } \xi = \left[\left(U, \phi, \begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix}\right)\right] \in T_pM$$

$\#$

<u>Remark</u>: In particular, we have $\dim T_pM = n$ with basis corresponds to $\left\{ \frac{\partial}{\partial x^j}\Big|_0 \right\}$ in local coordinates

$$\left(\text{where } \frac{\partial}{\partial x^j}\Big|_0 \in \mathcal{D}_0(\mathbb{R}^n) \text{ s.t. } \begin{pmatrix} \delta(x^1) \\ \vdots \\ \delta(x^n) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j\text{th place} \right)$$

<u>Convention</u>: If (U, ϕ) is a chart around p, and $(x^1, \cdots, x^n)$ are the corresponding coordinate functions

$$x^j : U \xrightarrow{\phi} \mathbb{R}^n \xrightarrow{\pi_j} \mathbb{R}.$$

We denote
$$\left(\frac{\partial}{\partial x^j}\right)_p (f) \overset{\text{def}}{=} \frac{\partial}{\partial x^j}(f \circ \phi^{-1})(\phi(0))$$

In this notation
$$L_\xi = \sum_{j=1}^{n} u^j \left(\frac{\partial}{\partial x^j}\right)_p \quad \text{for } \xi = [(U, \phi, u)] \in T_pM.$$

Hence $\left(\frac{\partial}{\partial x^j}\right)_p$ can be regraded as a vector in T_pM;

$\Rightarrow$ $\frac{\partial}{\partial x^j}$ is a vector field on $U \subset M$.

If $X^1, \cdots, X^n$ are smooth functions, then

$$X = \sum_{j=1}^{n} X^j \frac{\partial}{\partial x^j} \quad \text{is a vector field on } U$$

corresponds to

$$L_X : C^\infty(U) \to C^\infty(U) \quad \text{defined by}$$

$$(L_X f)(p) = \sum_{j=1}^{n} X^j(p) \left(\frac{\partial f}{\partial x^j}\right)_p .$$

Thm: The map $X \mapsto L_X$ is an isomorphism between the vector spaces $\Gamma(TM)$ and $\mathcal{D}(M)$, where $\mathcal{D}(M)$ = set of derivations δ on M defined by
(i) $\delta : C^\infty(M) \to C^\infty(M)$ linear;
(ii) $\delta(fg) = f\delta(g) + g\delta(f)$.

(Pf: Omitted.
Caution: Analog statement for complex manifold is not true, since we need to use cut-off functions to reduce it to coordinate systems.)

Note: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then $\delta_1 \circ \delta_2 \notin \mathcal{D}(M)$

Lemma: If $\delta_1, \delta_2 \in \mathcal{D}(M)$, then

$$\delta_1 \circ \delta_2 - \delta_2 \circ \delta_1 \in \mathcal{D}(M).$$

(Pf: Ex!)

Def: Let X, Y be vector fields on M. Then $[X,Y]$, the bracket of X & Y, is the vector field corresponding to the derivation $L_X \circ L_Y - L_Y \circ L_X$.

i.e. $$L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X.$$

Local formula for $[X,Y]$:

If $X = \sum_{i=1}^{n} X^i \frac{\partial}{\partial x^i}$, $Y = \sum_{j=1}^{n} Y^j \frac{\partial}{\partial x^j}$ in some local coordinates, then

$$L_X f = \sum_i X^i \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \quad L_Y L_X f = \sum_{i,j} Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} + Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i}$$

Similar formula for $L_X L_Y$.

$$\Rightarrow (L_X L_Y - L_Y L_X) f = \sum_i \sum_j \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i}$$

$$\Rightarrow \boxed{\begin{aligned} [X,Y] &= \sum_i Z^i \frac{\partial}{\partial x^i} \\ \text{where} \quad Z^i &= \sum_j \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \end{aligned}}$$

<u>Lemma</u> (<u>Jacobi Identity</u>) For vector fields X, Y, Z,

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

(Pf: Trivial)

1.6 <u>Vector Bundles and Tensors</u>

<u>Def</u>: Let E & B be 2 smooth manifolds and

$\pi : E \to B$ be a smooth map.

(π, E, B) is a <u>vector bundle of rank n</u>,

if
- π is surjective.
- $\exists$ open covering $(U_i)_{i \in \Lambda}$ of B, and diffeomorphisms $h_i : \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ such that $\forall\, x \in U_i$, $h(\pi^{-1}(x)) = \{x\} \times \mathbb{R}^n$

(hence $\pi^{-1}(x)$ can be regarded as a vector space.)

- and such that $\forall\, i, j \in \Lambda$, the diffeomorphism

$$h_i \circ h_j^{-1} : (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n$$

are of the form

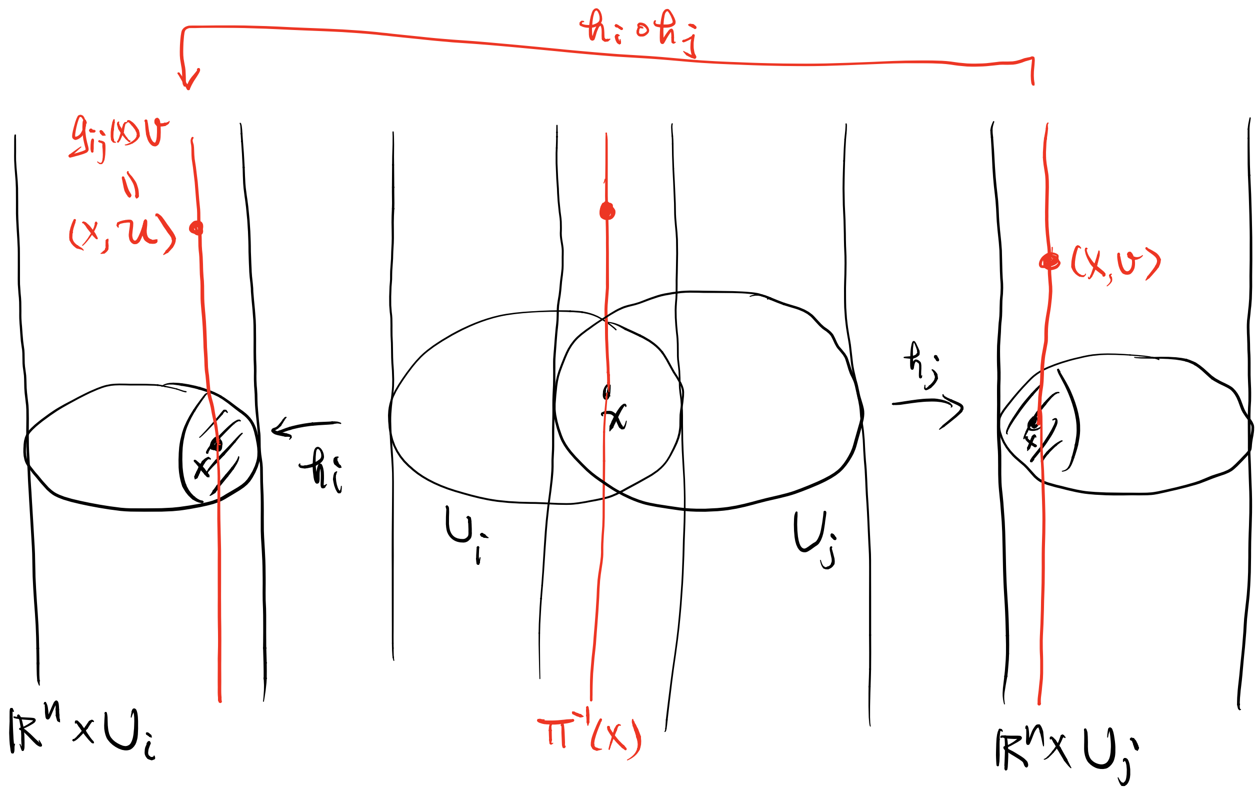
$$h_i \circ h_j^{-1}(x, v) = (x,\ g_{ij}(x) v)$$

where $\quad g_{ij} : U_i \cap V_j \to GL(n, \mathbb{R})$.

<u>Terminology</u>: E = <u>total space</u>, B = <u>base</u>

$\mathbb{R}^n$ or $\pi^{-1}(x)$ = <u>fibre</u>

h_i = <u>local trivialization</u>



eg: (Trivial Bundle): $\pi: M \times \mathbb{R}^n \to M$

$$(x, v) \longmapsto x$$

eg: Tangent bundle of M: $TM = \coprod_{p \in M} T_pM$

(exercise!)

Def: (a) A vector bundle of rank n, $\pi: E \to B$, is trivial

if $\exists$ a diffeomorphism

$$h: E \to B \times \mathbb{R}^n$$

s.t. $h: \pi^{-1}(x) \to \{x\} \times \mathbb{R}^n$ is a

vector space isomorphism.

(b) A (global) section of the bundle is a

smooth map $s: B \to E$ such that

$\pi \circ s = \mathrm{id}$

$$\begin{tikzcd} E \arrow[d, shift right, "\pi"'] \\ B \arrow[u, shift right, "s"'] \end{tikzcd}$$

eg: vector field $X \in \Gamma(M)$ $(= \Gamma(TM))$ is a section

of the tangent bundle TM.

Tensor product

Def: Let E, F be 2 finite dimensional vector space, then $E \otimes F$, the tensor product of E & F, is defined as the vector space, unique up to isomorphism, such that $\forall$ vector space G,

$$L(E \otimes F, G) \overset{\text{isom}}{\cong} L_2(E \times F, G)$$

(linear transformations from $E \otimes F$ to G) (bilinear maps from $E \times F$ to G)

Remark: $\exists$ a bilinear map $\otimes : E \times F \to E \otimes F$ such that if $\{e_i\}$ = basis of E and $\{f_i\}$ = basis of F,

then $\{e_i \otimes f_j\}_{i,j}$ is a basis of $E \otimes F$.

Hence for $u = \sum_i a^i e_i \in E$ and $v = \sum_j b^j f_j \in F$,

then $u \otimes v = \sum_{i,j} a^i b^j e_i \otimes f_j$.

Facts: (1) If $E^* = $ dual of $E = L(E, \mathbb{R})$

$F^* = $ dual of F

then $$E^* \otimes F^* \simeq L_2(E \times F, \mathbb{R})$$
$$\cong L(E \otimes F, \mathbb{R}) = (E \otimes F)^*$$

(by $\alpha \otimes \beta \longmapsto \alpha \otimes \beta (u \otimes v) = \alpha(u)\beta(v)$)

(2) If $\alpha \in L(E, E')$ & $\beta \in L(F, F')$

(E, E', F, F' are finite dim'l vector spaces)

then one can define

$$\alpha \otimes \beta \in L(E \otimes F, E' \otimes F')$$

by $(\alpha \otimes \beta)(u \otimes v) \stackrel{\text{def}}{=\!=} \alpha(u) \otimes \beta(v)$.

(3) Given a vector bundle E (with fibers E_x, $x \in M$), one can define the vector bundle E^*, $\otimes^p E$ (with fibers E_x^* and $\otimes^p E_x$ respectively)

(4) Given 2 vector bundles E, F (with fibers E_x, F_x) with the <u>same base manifold</u> M, we can define the vector bundle $E \otimes F$ over M with fiber $E_x \otimes F_x$.

eg: Starting from TM, we can define the <u>cotangent bundle</u>

T^*M of M, and the (p,q)-tensor bundle

$$\left(\bigotimes^p TM\right) \otimes \left(\bigotimes^q T^*M\right) \quad \text{of } M.$$

Def: A (p,q)-tensor (field), or more precisely p times contravariant & q times covariant tensor, on M is a smooth section of the bundle $\left(\bigotimes^p TM\right) \otimes \left(\bigotimes^q T^*M\right)$.

Note: For $f: M \to \mathbb{R}$ smooth, we can define

$$df \in \Gamma(T^*M) \quad \text{by} \quad df(X) = L_X f$$
$$= Xf \,, \; \forall X \in \Gamma(M))$$

Then $\{dx^j\}_{j=1}^n$ is a dual (local) basis to

$$\left\{\frac{\partial}{\partial x^j}\right\}_{j=1}^n \quad \text{since} \quad dx^j\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial x^j}{\partial x^i} = \delta_i^j$$

at each point in a coordinate system with coordinate functions $(x^1, \cdots, x^n)$.

Therefore

$$\left\{ \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_q} \right\}$$

forms a local basis for $\left(\bigotimes^p TM\right) \otimes \left(\bigotimes^q T^*M\right)$

$\Rightarrow$ in coordinates, a (p,q)-tensor (field) can be written as

$$T = T^{j_1 \cdots j_p}_{i_1 \cdots i_q} \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_p}} \otimes dx^{i_1} \times \cdot \; dx^{i_q}$$