MATH2020A Tutorial 9

1. Find out the potential functions and evaluate the line integral.

$$\int_{(1,0,0)}^{(2,2,2)} (x^2 - 2yz)dx + (y^2 - 2xz)dy + (z^2 - 2xy)dz$$

Let's assume $df(x, y, z) = (x^2 - 2yz)dx + (y^2 - 2xz)dy + (z^2 - 2xy)dz$. This means

$$\frac{\partial f}{\partial x} = x^2 - 2yz$$
$$\frac{\partial f}{\partial y} = y^2 - 2xz$$
$$\frac{\partial f}{\partial z} = z^2 - 2xy$$

For the first equation, we have

$$f(x, y, z) = \int (x^2 - 2yz)dx = \frac{x^3}{3} - 2xyz + C(y, z)$$

where $C(y, z)$ is a function only depending on y, z. Similarly, we have other two result

$$f(x, y, z) = \int (y^2 - 2xz)dx = \frac{y^3}{3} - 2xyz + C(x, z)$$

$$f(x, y, z) = \int (z^2 - 2xy)dx = \frac{z^3}{3} - 2xyz + C(x, y)$$

So the only way to choose f is

$$f(x, y, z) = \frac{x^3 + y^3 + z^3}{3} - 2xyz + C$$

to satisfies above three equations with C a constant free to choose. Of course, we can choose $C = 0$. So by this protentional function, we get our result

$$\begin{aligned} & \int_{(1,0,0)}^{(2,2,2)} (x^2 - 2yz)dx + (y^2 - 2xz)dy + (z^2 - 2xy)dz \\ = & f(2,2,2) - f(1,0,0) \\ = & \frac{8 \times 3}{3} - 2 \times 2 \times 2 \times 2 - \frac{1}{3} \\ = & \frac{23}{3} \end{aligned}$$

2. Find out the potential functions and evaluate the line integral.

$$\int_{(1,0,0)}^{(2,2,2)} \frac{xdx + ydy + zdz}{1 + (x^2 + y^2 + z^2)^2}$$

Notice we have $xdx + ydy + zdy$ in our integrals. This is exactly the gradient of the function $\frac{x^2+y^2+z^2}{2}$, i.e. $d(\frac{x^2+y^2+z^2}{2}) = xdx + ydy + zdz$. So we can guess the potential function will take the form

$$f(x, y, z) = g(x^2 + y^2 + z^2)$$

where $g(w) : \mathbb{R} \to \mathbb{R}$ is just a function on real line. Take differential, we will get

$$df(x, y, z) = g'(x^2+y^2+z^2)d(x^2+y^2+z^2) = 2g'(x^2+y^2+z^2)(xdx+ydy+zdz)$$

We wish to have

$$df(x, y, z) = \frac{xdx + ydy + zdz}{1 + (x^2 + y^2 + z^2)^2}$$

So we only need to make sure

$$2g'(x^2 + y^2 + z^2) = \frac{1}{1 + (x^2 + y^2 + z^2)^2}$$

Or in another form,

$$2g'(w) = \frac{1}{1 + w^2}$$

So we get

$$g(w) = \frac{1}{2} \arctan(w) + C$$

We just take C as 0 and we have

$$f(x,y,z)=\frac{1}{2}\arctan(x^2+y^2+z^2)$$

For our integration, we have

$$\begin{aligned}
& \int_{(1,0,0)}^{(2,2,2)} \frac{xdx+ydy+zdz}{1+(x^2+y^2+z^2)^2} \\
= & f(2,2,2)-f(1,0,0) \\
= & \frac{1}{2}(\arctan(12)-\arctan(1)) \\
= & \frac{\arctan(12)}{2}-\frac{\pi}{8}
\end{aligned}$$

3. Using Green's Formula to calculate the following integration.

$$\oint_C xy^2dy-x^2ydx \quad C \text{ is circle } x^2+y^2=1 \text{ with counterclockwise orientation}$$

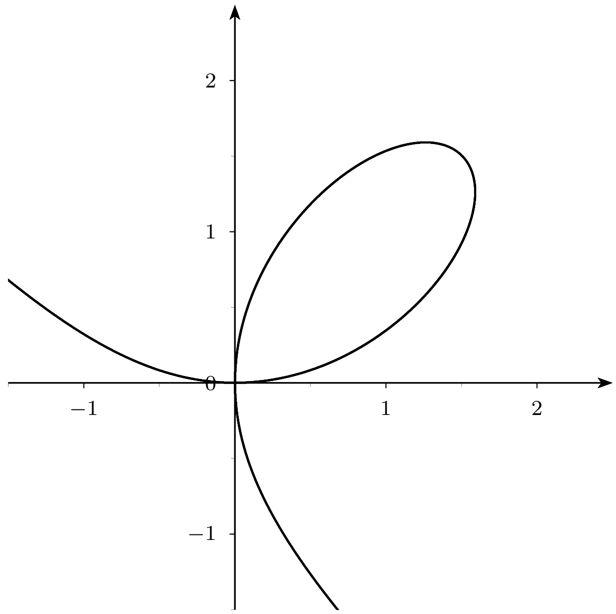
Just apply Green's formula and we also want to use polar coordinate to find out the final result.

$$\begin{aligned}
\oint_C xy^2dy-x^2ydx &= \iint\limits_{R:=\{(x,y):x^2+y^2\le 1\}} y^2-(-x^2)dxdy \\
&= \iint\limits_{R}(x^2+y^2)dxdy \\
&= \int_0^{2\pi}\int_0^1 r^2rdrd\theta \\
&= \int_0^{2\pi}\frac{1}{4}d\theta \\
&= \frac{\pi}{2}
\end{aligned}$$

4. Find out the area enclosed by folium of Descartes using Green's Theorem which defined by

$$C=\{(x,y):x^3+y^3=3xy\}$$

The following is a picture of folium of Descartes.



With Green's formula, we know that area enclosed by a curve can be compute by following formulas

$$\text{Area}(R) = \iint_R dxdy = \oint_C xdy = \oint_C -ydx = \frac{1}{2}\oint_C xdy - ydx$$

First, we need to find out a parameter for this curve. Choose $y = tx$ where t is our parameter. Then we have

$$x^3 + t^3x^3 = 3tx^2$$

which implies

$$x = \frac{3t}{1+t^3}$$

and hence

$$y = tx = \frac{3t^3}{1+t^3}$$

. The curve in first quadrant can be written as

$$r(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right), t \in [0, \infty)$$

Hence

$$
\begin{aligned}
\iint\limits_R dxdy &= \oint_C -ydx = -\int_0^\infty \frac{3t^2}{1+t^3} d\frac{3t}{1+t^3} \\
&= -\int_0^\infty \frac{9t^2(1-2t^3)}{(1+t^3)^3} dt = \int_0^\infty \frac{3(2s-1)}{(1+s)^3} ds \quad (x = t^3) \\
&= \int_0^\infty \frac{6}{(1+s)^2} - \frac{9}{(1+s)^3} ds = \left[-\frac{6}{1+s} + \frac{9}{2(1+s)^2} \right]_0^\infty \\
&= \frac{3}{2}
\end{aligned}
$$

5. Suppose u is a harmonic function in a domain D (We also assume R is simply connected here), i.e. u satisfies $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$. Show that for any curve $C : r(t)$ in this domain, we have

$$\oint_C \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx = 0$$

And using this fact to proof the average equality of harmonic function, i.e., show that

$$u(x_0, y_0) = \frac{1}{2\pi r} \int\limits_{C:=\{(x,y):(x-x_0)^2+(y-y_0)^2=r^2\}} u(x, y)dxdy \text{ (Average of } u \text{ on circle)}$$

For the first integral, we just apply Green's formula to get

$$\oint_C \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx = \iint_R \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} dxdy = \iint_R 0 dxdy = 0$$

just by definition of harmonic function.

Now we choose C to be a circle $\{(x, y) : (x - x_0)^2 + (y - y_0)^2 = r^2\}$, i.e., we choose

$$C : r(t) = (x_0 + r \cos t)\mathbf{i} + (y_0 + r \sin t)\mathbf{j}$$

Integrate on this curve will give us

$$0 = \oint_C \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx = \int_0^{2\pi} (\frac{\partial u}{\partial x} r \cos t + \frac{\partial u}{\partial y} r \sin t) dt$$

If we choose function $I(r)$ to denote the average of u on the circle $\{(x, y) : (x - x_0)^2 + (y - y_0)^2 = r^2\}$, i.e., we write

$$I(r) = \frac{1}{2\pi r} \oint_C u(x, y) ds$$

We still use parameter of this circle, i.e., we have

$$I(r) = \frac{1}{2\pi r} \int_0^{2\pi} u(r \cos t, r \sin t) r dt = \frac{1}{2\pi} \int_0^{2\pi} u(r \cos t, r \sin t) dt$$

Thus, take derivative and we get

$$I'(r) = \frac{1}{2\pi} \int_0^{2\pi} \left(\frac{\partial u}{\partial x}(r \cos t, r \sin t) \cos t + \frac{\partial u}{\partial y}(r \cos t, r \sin t) \sin t \right) dt$$

But from above computational, we have

$$\int_0^{2\pi} \frac{\partial u}{\partial x} \cos t + \frac{\partial u}{\partial y} \sin t dt = 0$$

Hence, we have $I'(r) = 0$. This means, $I(r)$ is just a constant function, does not depending on r. So we letting $r \to 0$, we get $\lim_{r \to 0} I(r) = u(x_0, y_0)$ because of the average converging to the center point of the circle as u is a continuous function. And using $I(r)$ is constant everywhere we know that $I(r) = u(x_0, y_0)$. This is exactly the solution.