Proof of Divergence Thm

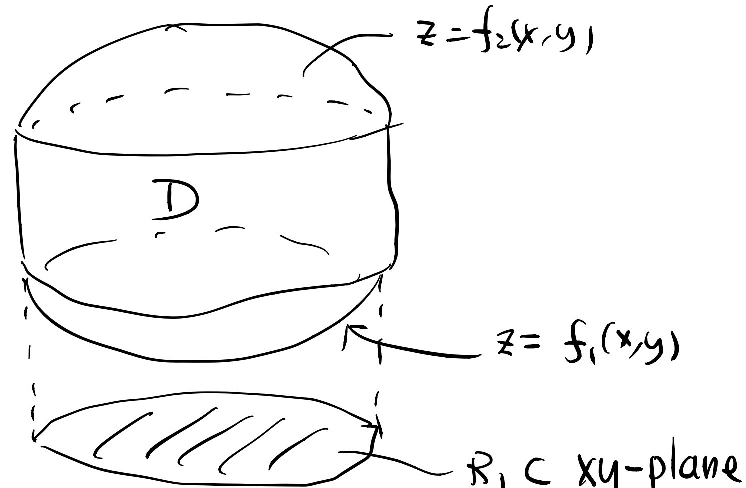
Same as Green's Thm, we'll prove only the case of special domain D which is of type I, II, and III:

$$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R_1,\ f_1(x,y) \le z \le f_2(x,y)\} \quad (\text{type I})$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in R_2,\ g_1(y,z) \le x \le g_2(y,z)\} \quad (\text{type II})$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (x,z) \in R_3,\ h_1(x,z) \le y \le h_2(x,z)\} \quad (\text{type III})$$

eg: type I domain:



And also as in the proof of Green's Thm,

for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

we'll prove 3 equalities in the following which combine to give the divergence thm:

$$\iint_S M\hat{i} \cdot \hat{n}\, d\sigma = \iiint_D \frac{\partial M}{\partial x}\, dV \quad (\text{by type II})$$

$$\iint_S N\hat{j} \cdot \hat{n}\, d\sigma = \iiint_D \frac{\partial N}{\partial y}\, dV \quad (\text{by type III})$$

$$\iint_{S} L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_{D} \frac{\partial L}{\partial z}\,dV \qquad \text{(by type I)}$$

The proofs are similar, we'll prove only the last one:

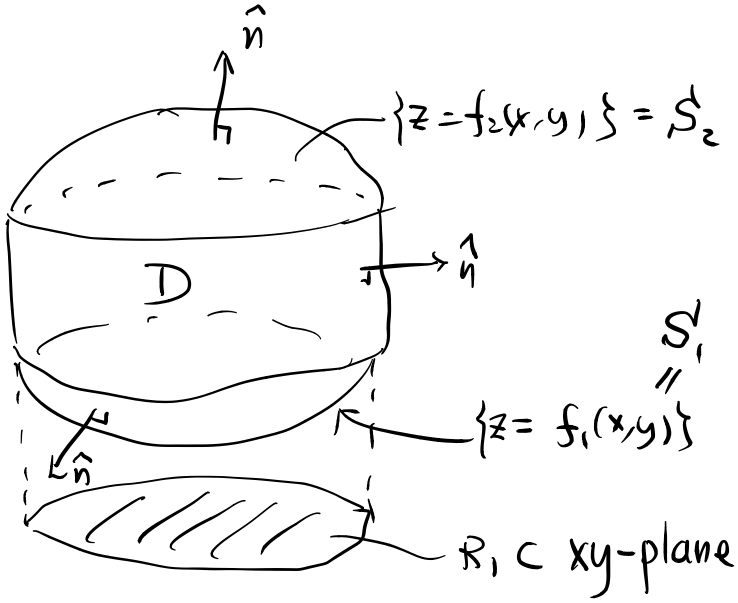
$$\iint_{S} L\hat{k}\cdot\hat{n}\,d\sigma = \iiint_{D} \frac{\partial L}{\partial z}\,dV$$

By Fubini's Thm,

$$\text{R.H.S.} = \iiint_{D} \frac{\partial L}{\partial z}\,dV = \iint_{R_1}\left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial L}{\partial z}\,dz\right]dx\,dy$$

$$= \iint_{R_1}\left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right]dx\,dy$$

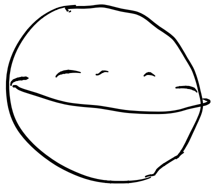
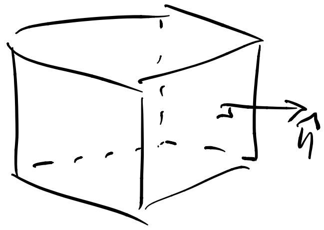
For the L.H.S., we note by definition of type I domain, the boundary surface S of D can be written as $S = S_1 \cup S_2 \cup S_3$,



where S_1 = graph of $f_1 = \{(x,y,f_1(x,y))\} = \{z = f_1(x,y)\}$

S_2 = graph of $f_2 = \{(x,y,f_2(x,y))\} = \{z = f_2(x,y)\}$

S_3 = a vertical surface (which could be empty) between S_1 & S_2.

( $S_3 = \text{empty}$; 

$S_3 =$ union of 4 vertical squares)

cube

Hence

$$\text{L.H.S.} = \iint_S L\,\hat{k}\cdot\hat{n}\,d\sigma = \iint_{S_1} L\hat{k}\cdot\hat{n}\,d\sigma + \iint_{S_2} L\hat{k}\cdot\hat{n}\,d\sigma$$

$$+ \iint_{S_3} \cancelto{0}{L\hat{k}\cdot\hat{n}}\,d\sigma$$

(since $\hat{n}$ of a vertical surface is <u>horizontal</u>)

Now on the upper surface $S_2 = \{ z = f_2(x,y) \}$,

the outward normal $\hat{n}$ is <u>upward</u> (in the sense that $\hat{n}\cdot\hat{k} > 0$).

Note that the parametrization

$$(x,y) \longmapsto \vec{r}(x,y) = x\,\hat{i} + y\,\hat{j} + f_2(x,y)\,\hat{k}$$

has

$$\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f_2}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \dfrac{\partial f_2}{\partial y}\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \dfrac{\partial f_2}{\partial x} \\ 0 & 1 & \dfrac{\partial f_2}{\partial y} \end{vmatrix} = -\frac{\partial f_2}{\partial x}\hat{i} - \frac{\partial f_2}{\partial y}\hat{j} + \hat{k}$$

+ve $\Rightarrow$ $\vec{r}_x \times \vec{r}_y$ is upward

Hence $$\hat{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$$

and $\quad \hat{k}\cdot\hat{n} = \dfrac{1}{|\vec{r}_x \times \vec{r}_y|}$

Therefore

$$\iint_{S_2} L\hat{k}\cdot\hat{n}\, d\sigma = \iint_{R_1} L(x,y,f_2(x,y)) \overbrace{\frac{1}{|\vec{r}_x\times\vec{r}_y|}}^{\hat{k}\cdot\hat{n}} \overbrace{|\vec{r}_x\times\vec{r}_y|\, dA}^{d\sigma}$$

$$= \iint_{R_1} L(x,y,f_2(x,y))\, dx\,dy$$

Similarly, note that the outward normal at S_1 (lower surface) is <u>downward</u> (i.e. $\hat{n}\cdot\hat{k} < 0$), we have

$$\hat{n} = -\frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x\times\vec{r}_y|}, \quad \text{where } \vec{r}(x,y) = x\hat{i} + y\hat{j} + f_1(x,y)\hat{k}$$

$$\Rightarrow \quad \hat{k}\cdot\hat{n} = -\frac{1}{|\vec{r}_x\times\vec{r}_y|} \qquad \text{(Check!)}$$

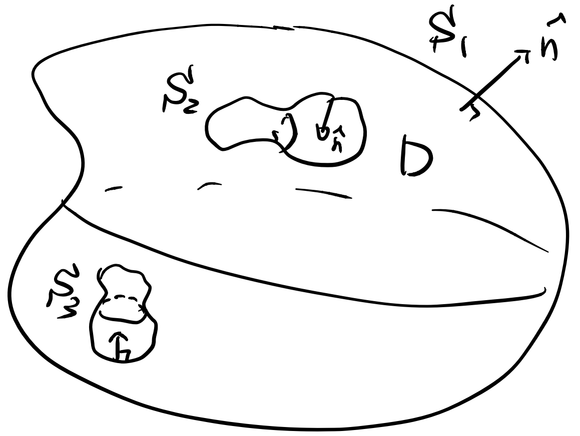
Hence $\displaystyle\iint_{S_1} L\hat{k}\cdot\hat{n}\, d\sigma = -\iint_{R_1} L(x,y,f_1(x,y))\, dx\,dy$

Therefore $\displaystyle\iint_{S} L\hat{k}\cdot\hat{n}\, d\sigma = \iint_{R_1} [L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))]\, dx\,dy$

$$= \iiint_D \frac{\partial L}{\partial z}\, dV \qquad \#$$

Note: Similar to Green's Thm, the Divergence Thm is also hold for solid region with finitely many holes inside:

$$\iiint_D \vec{\nabla} \cdot \vec{F} \, dV = \sum_{i=1}^{n} \iint_{S_i} \vec{F} \cdot \hat{n} \, d\sigma$$



for $\hat{n}$ = outward normal with respect to D.

Note: Physical meaning of $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$ in $\mathbb{R}^3$

= flux density (by the divergence theorem)

Unified treatment of Green's, Stokes', and Divergence Theorems

Stokes' Thm in notations of differential forms (in $\mathbb{R}^3$)

Working definition of differential forms

(1) A differential 1-form (or simply 1-form)

is a linear combination of the symbols dx, dy & dz:

$$\boxed{\omega = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz}$$

with coefficients $\omega_1, \omega_2, \omega_3$ are functions on $\mathbb{R}^3$.

eg: The total differential of a smooth function f

is a differential 1-form:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

(2) Wedge product: Let "$\wedge$" be an operation such that

$$\begin{cases} dx\wedge dx = dy\wedge dy = dz\wedge dz = 0 \\ dx\wedge dy = -dy\wedge dx \\ dy\wedge dz = -dz\wedge dy \\ dz\wedge dx = -dx\wedge dz \end{cases}$$

and satisfies other usual rules in arithmetic.

i.e. If $\omega = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz$

$\eta = \eta_1\, dx + \eta_2\, dy + \eta_3\, dz$

then we have

$$\omega \wedge \eta = (\omega_1 dx + \omega_2 dy + \omega_3 dz) \wedge (\eta_1 dx + \eta_2 dy + \eta_3 dz)$$

$$= \omega_1 \cancel{dx \wedge \eta_1 dx}^{\,0} + \omega_2 dy \wedge \eta_1 dx + \omega_3 dz \wedge \eta_1 dx$$
$$+ \omega_1 dx \wedge \eta_2 dy + \omega_2 \cancel{dy \wedge \eta_2 dy}^{\,0} + \omega_3 dz \wedge \eta_2 dy$$
$$+ \omega_1 dx \wedge \eta_3 dz + \omega_2 dy \wedge \eta_3 dz + \omega_3 \cancel{dz \wedge \eta_3 dz}^{\,0}$$

$$= (\omega_1\eta_2 - \omega_2\eta_1)\, dx \wedge dy$$
$$+ (\omega_2\eta_3 - \omega_3\eta_2)\, dy \wedge dz$$
$$+ (\omega_3\eta_1 - \omega_1\eta_3)\, dz \wedge dx$$

$\therefore$

$$\boxed{\begin{aligned}\omega \wedge \eta = {} & (\omega_2\eta_3 - \omega_3\eta_2)\, dy \wedge dz \\ & + (\omega_3\eta_1 - \omega_1\eta_3)\, dz \wedge dx \\ & + (\omega_1\eta_2 - \omega_2\eta_1)\, dx \wedge dy\end{aligned}}$$

- Linear combinations of $dy \wedge dz$, $dz \wedge dx$ & $dx \wedge dy$ are called <u>differential 2-forms</u> (on $\mathbb{R}^3$)

$$\boxed{\zeta = \zeta_1\, dy \wedge dz + \zeta_2\, dz \wedge dx + \zeta_3\, dx \wedge dy}$$

Similarly, if ω is a 1-form and ζ is a 2-form

then we can define $\omega \wedge \zeta$

eg: If $\omega = dx$, $\zeta = dy \wedge dz$

then $\omega \wedge \zeta = dx \wedge dy \wedge dz$

Note that we insist on the anti-commutativity of wedge product, we have

$$
\begin{aligned}
dx \wedge dy \wedge dz &= -dy \wedge dx \wedge dz \\
&= dy \wedge dz \wedge dx \\
&= -dz \wedge dy \wedge dx \\
&= dz \wedge dx \wedge dy \\
&= -dx \wedge dz \wedge dy
\end{aligned}
$$

And $dx \wedge dx \wedge dy = \cdots = 0$ whenever one of the dx, dy, dz repeated.

Hence, as $\dim \mathbb{R}^3 = 3$, all "linear combinations" of "3-forms" are just

$$f\, dx \wedge dy \wedge dz$$

which is called a differential 3-form (also called a volume form if $f > 0$)

Note: It is convenient to call smooth functions f the differential 0-form.

Summary (on $\mathbb{R}^3$)

0-form : f

1-form : $\omega_1 dx + \omega_2 dy + \omega_3 dz$

2-form : $\xi_1 dy \wedge dz + \xi_2 dz \wedge dx + \xi_3 dx \wedge dy$

3-form : $g\, dx \wedge dy \wedge dz$.

where, f, g, ω_i, ξ_i are (smooth) functions

Note: On can certainly define k-form for any $k \geq 0$. But in $\mathbb{R}^3$, k-forms are zero for $k > 3$:

$dx^i \wedge dx \wedge dy \wedge dz = 0$, where $dx^i = dx, dy$, or dz.

Change of Variables Formula: ($\mathbb{R}^2$)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\Rightarrow \begin{cases} dx = x_u\, du + x_v\, dv \\ dy = y_u\, du + y_v\, dv \end{cases}$$

$$\Rightarrow dx \wedge dy = (x_u\, du + x_v\, dv) \wedge (y_u\, du + y_v\, dv)$$

$$= (x_u y_v - x_v y_u)\, du \wedge dv$$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du \wedge dv$$

$$\boxed{dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv}$$

Jacobian determinant.

Hence naturally

$$\boxed{\iint f(x,y)\, dx \wedge dy = \iint f(x(u,v), y(u,v))\, \frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv}$$

Compare with

$$\boxed{\iint f(x,y)\, dx\, dy = \iint f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv}$$