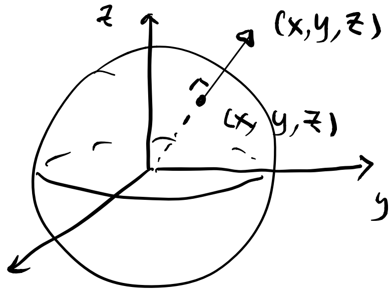
To integrate vector fields over a surface, we need

> **Def 17** (Orientation of a surface in $\mathbb{R}^3$)
>
> A surface S is orientable if one can define a unit normal vector field continuously at every point of S

eg 57 : (i) $S^2 = \{x^2+y^2+z^2=1\}$



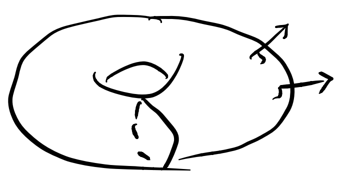
$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= x\hat{i}+y\hat{j}+z\hat{k} \quad \text{on } S^2$$

is a continuous unit normal vector field on S^2

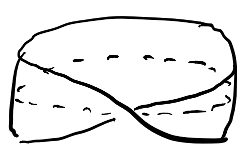
$\Rightarrow$ S^2 is orientable.

(ii)



Torus is orientable

(iii)



Möbius strip is not orientable
(usually refered as surface of one side)

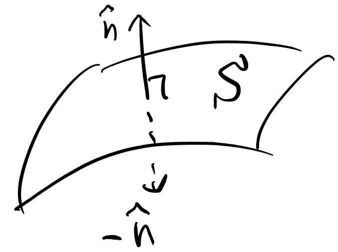
Remark: Parametric surface $S : \vec{r}(u,v)$ are alway orientable.

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$ is a continuous unit normal vector field on S

(since $|\vec{r}_u \times \vec{r}_v| > 0$)

Terminology

Given a connected orientable surface $S \subset \mathbb{R}^3$, there are two ways to assign the continuous unit normal vector field



Suppose S is orientable and we have chosen one continuous unit normal vector field $\hat{n}$. Then

Def 18: We said that a parametrization $\vec{r}(u,v)$ of S is compatible with the orientation of S given by the unit normal vector field $\hat{n}$,

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$

Def 19: Let S be orientable with unit normal $\hat{n}$.

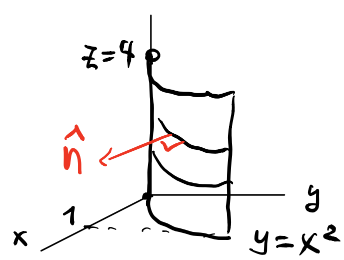
Let $\vec{F}$ be a vector field on S.

Then the flux of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n} \, d\sigma$$

eg 59 $\quad S: y = x^2 \quad 0 \le x \le 1$

$\qquad\qquad\qquad\qquad 0 \le z \le 4$



with $\hat{n}$ given by the

natural parametrization

$$\vec{r}(x,z) = x\hat{i} + x^2\hat{j} + z\hat{k}$$

$$\begin{cases} \vec{r}_x = \hat{i} + 2x\hat{j} \\ \vec{r}_z = \hat{k} \end{cases} \Rightarrow \vec{r}_x \times \vec{r}_z = (\hat{i} + 2x\hat{j}) \times \hat{k} = 2x\hat{i} - \hat{j}$$

$$\Rightarrow \hat{n} = \frac{2x\hat{i} - \hat{j}}{\sqrt{4x^2+1}}$$

Let $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$.

Find $\iint_S \vec{F}\cdot\hat{n}\,d\sigma$.

Soln: $$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \int_0^4\int_0^1 \underbrace{(yz\hat{i} + x\hat{j} - z^2\hat{k})}_{\vec{F}} \cdot \underbrace{\frac{2x\hat{i} - \hat{j}}{\cancel{\sqrt{4x^2+1}}}}_{\hat{n}} \underbrace{\cancel{\sqrt{4x^2+1}}\,dx\,dz}_{d\sigma}$$

$$= \int_0^4\int_0^1 (2x^3z - x)\,dx\,dz \quad \text{(check!)}$$

$$= 2 \quad \text{(Check!)}$$

Remark: $$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{(u,v)} \vec{F}(\vec{r}(u,v)) \cdot \frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\,|\vec{r}_u\times\vec{r}_v|\,du\,dv$$

$$= \iint_{(u,v)} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u\times\vec{r}_v)\,du\,dv.$$

Thm 12 (Stokes' Theorem)

Let S be a piecewise smooth oriented surface with piecewise smooth boundary C (including the case that C is a union of finitely many curves). Let

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \quad \text{be a } C^1 \text{ vector field.}$$

Suppose C is oriented anti-clockwisely with respect to the unit normal vector field $\hat{n}$ on S. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot \hat{n}\, d\sigma$$

$$= \iint_S (\vec{\nabla}\times\vec{F})\cdot \hat{n}\, d\sigma$$

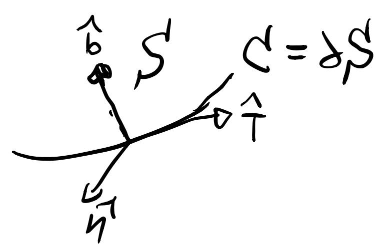
Hence (i) if $C = C_1 \cup \cdots \cup C_k$, then it means

$$\sum_{i=1}^{k} \oint_{C_i} \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma .$$

(ii) "C is oriented anti-clockwisely with respect to the unit normal vector field $\hat{n}$" means that

we choose the direction of C such that its (unit) tangent vector $\hat{T}$ satisfies

$$\boxed{\hat{b} = \hat{n}\times\vec{T} \text{ pointing toward the surface } S.}$$



Note: $\hat{b}$ is a (unit) tangent vector to S and normal to C and pointing toward S. Then

$$\hat{T} = \hat{b} \times \hat{n} .$$