Pf of Green's Thm (tangential form)

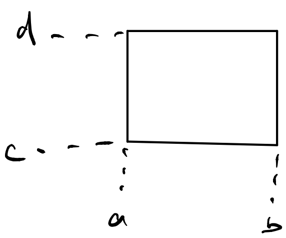
Recall : A region R is of special type :

type (1) : If $R = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$
for some continuous functions $g_1(x)$ & $g_2(x)$.

type (2) : If $R = \{(x,y) : h_1(y) \leq x \leq h_2(y),\ c \leq y \leq d\}$
for some continuous functions $h_1(y)$ & $h_2(y)$.

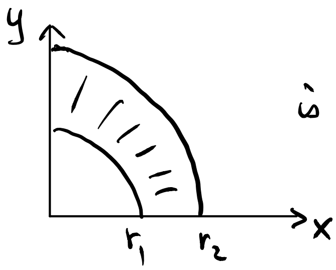
Now : If R is both type (1) & type (2), it said to be simple.

eg 48 (i)



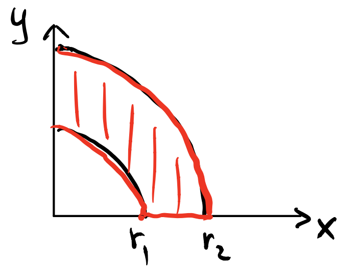
rectangle is simple

(ii)

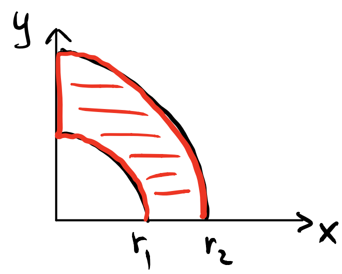


is simple

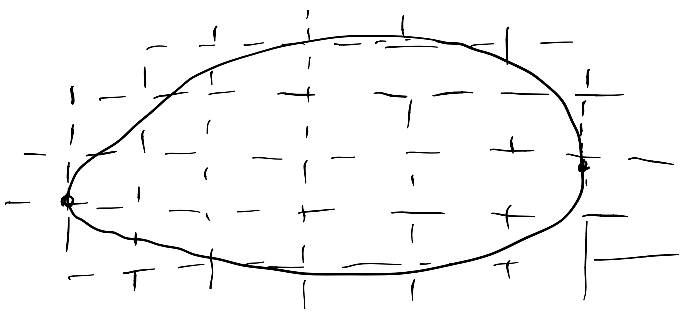
type (1) : Yes



type (2) : Yes



(iii)



2 intersection at most

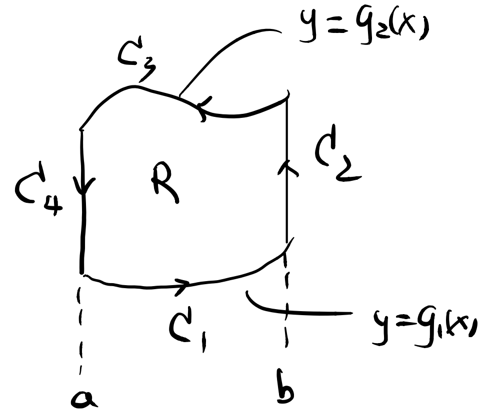
2 intersection at most

$\forall a \in \mathbb{R}$: $\#\{\partial R \cap \{x = a\}\} \leq 2$
$\#\{\partial R \cap \{y = a\}\} \leq 2$
$\Rightarrow$ simple
(provided ∂R is piecewise smooth)

Pf of Green's Thm for Simple Region

By definition, R is of type (I) and can be written as

$$R = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$



Let denote the components of the boundary of R by C_1, C_2, C_3, C_4 as in the figure (Note that C_2 and/or C_4 could just be a point.)

Then $\partial R = C_1 + C_2 + C_3 + C_4$ as oriented curve (using "+" instead of "$\cup$" to denote the orientation)

Now $C_1 = \{y = g_1(x)\}$ can be parametrized by

$$(x,y) = \vec{r}(t) = (t, g_1(t)) \quad , \quad a \leq t \leq b$$

with correct orientation.

$$\therefore \int_{C_1} M\,dx = \int_a^b M(t, g_1(t))\,dt$$

Similarly "$-C_3$" can be parametrized by

$$\vec{r}(t) = (t, g_2(t)) \quad , \quad a \leq t \leq b$$

with correct orientation

$$\therefore \int_{-C_3} M\,dx = \int_a^b M(t, g_2(t))\,dt$$

$$\Rightarrow \int_{C_3} M\,dx = -\int_{-C_3} M\,dx = -\int_a^b M(t, g_2(t))\,dt$$

For $C_2 = \{x=b\}$, it can be parametrized by

$$\vec{r}(t) = (b, t), \quad g_1(b) \le t \le g_2(b)$$

with correct orientation

$$\Rightarrow \int_{C_2} M\,dx = 0 \qquad \left(\text{since } \frac{dx}{dt} = 0\right)$$

Similarly $\int_{C_4} M\,dx = -\int_{-C_4} M\,dx = 0$.

Hence $\oint_{\partial R} M\,dx = \sum_{i=1}^{4} \int_{C_i} M\,dx$

$$= \int_a^b [M(t, g_1(t)) - M(t, g_2(t))]\,dt$$

$$\left(= \int_a^b [M(x, g_1(x)) - M(x, g_2(x))]\,dx \right)$$

On the other hand, Fubini's Thm $\Rightarrow$

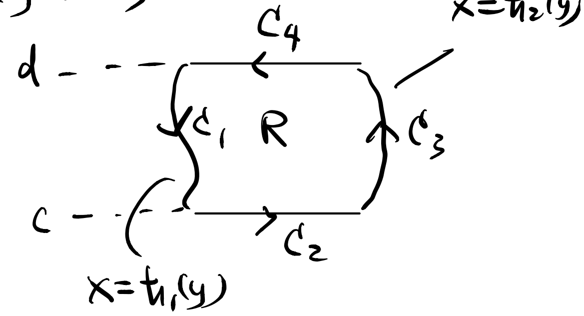
$$\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} -\frac{\partial M}{\partial y}\,dy \right] dx$$

$$= \int_a^b -[M(x, g_2(x)) - M(x, g_1(x))]\,dx$$

$$= \oint_{\partial R} M\,dx$$

Since R is also type (2), R can be written as

$R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$



$$\oint_{\partial R} N\,dy = -\int_c^d N(h_1(t),t)\,dt + 0 + \int_c^d N(h_2(t),t)\,dt + 0$$

$$= \int_c^d [N(h_2(t),t) - N(h_1(t),t)]\,dt$$

$$= \int_c^d [N(h_2(y),y) - N(h_1(y),y)]\,dy$$

$$= \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}(x,y)\,dx \right] dy$$

$$= \iint_R \frac{\partial N}{\partial x}\,dA$$

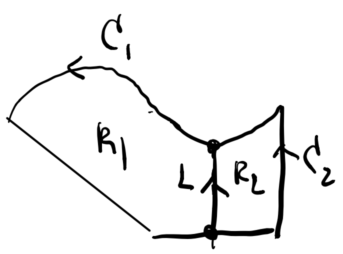
All together
$$\oint_{\partial R} (M\,dx + N\,dy) = \iint_R -\frac{\partial M}{\partial y}\,dA + \iint_R \frac{\partial N}{\partial x}\,dA$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

<u>Proof of Green's Thm for</u>

R = finite union of simple regions with intersections only along some boundary line segments, and those line segments touch only at the end points at most

[eg:



R_1, R_2 = simple

but $R = R_1 \cup R_2 \neq$ simple

$\partial R_1 = C_1 + L$

$\partial R_2 = C_2 - L$ (L :)

with anti-clockwise orientation

and $\partial R = C_1 + C_2$]

By assumption $R = \bigcup R_i$ finite union s.t.

- R_i are simple and
- $R_i \cap R_j$ = line segment of a common boundary portion denoted by L_{ij} $(i \neq j)$ (may be empty)

Then
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$
$$= \sum_i \oint_{\partial R_i} M\,dx + N\,dy \quad \left(\begin{array}{c}\text{by Green's Thm} \\ \text{for simple region}\end{array}\right)$$

Denote C_i = the part of ∂R_i with no intersection with any other R_j (except at the end points)

Then
$$\partial R_i = C_i + \sum_{\substack{j \\ (j \neq i)}} L_{ij}$$

where L_{ij} is oriented according to the anti-clockwise orientation of ∂R_i

Hence
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \oint_{C_i + \sum_{\substack{j \\ (j \neq i)}} L_{ij}} M\,dx + N\,dy$$
$$= \sum_i \int_{C_i} M\,dx + N\,dy + \sum_i \int_{\sum_{\substack{j \\ (j \neq i)}} L_{ij}} M\,dx + N\,dy$$

Note that, as C_i is not a common boundary of any other R_j,

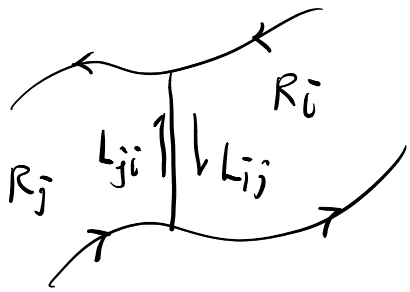
$$\sum_i C_i = \partial R$$

$$\therefore \quad \sum_i \int_{C_i} M\,dx + N\,dy = \oint_{\partial R} M\,dx + N\,dy$$

Finally, we have

$$L_{ji} = -L_{ij}$$

as R_i & R_j are located on the two different sides of the common boundary.



$$\sum_i \int_{\substack{\sum_j L_{ij} \\ (j\neq i)}} M\,dx + N\,dy = \sum_i \sum_{\substack{j \\ (j \neq i)}} \int_{L_{ij}} M\,dx + N\,dy$$

$$= \sum_{\substack{i,j \\ i \neq j}} \int_{L_{ij}} M\,dx + N\,dy$$

$$= \sum_{i<j} \int_{L_{ij}} M\,dx + N\,dy + \sum_{\underset{j}{\cancel{i}} > \underset{i}{\cancel{j}}} \int_{L_{\underset{j}{\cancel{i}}\underset{i}{\cancel{j}}}} M\,dx + N\,dy$$

$$= \sum_{i<j} \left(\int_{L_{ij}} M\,dx + N\,dy + \int_{L_{ji}} M\,dx + N\,dy \right)$$

$$= \sum_{i<j} \left(\int_{L_{ij}} M\,dx + N\,dy + \int_{-L_{ij}} M\,dx + N\,dy \right) = 0$$

This 2^{nd} case basically include almost all situations in the level of Advanced Calculus.

The proof of general case needs "analysis" and will be omitted here. #

$$\boxed{\text{Def 12: The \underline{divergence} of } \vec{F} = M\hat{i} + N\hat{j} \text{ is defined to be} \quad \operatorname{div}\vec{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}}$$

Note: $\operatorname{div}\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$

$$= \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{n}\, ds$$

$\underset{=}{\text{(called)}}$ "flux density"

Notation: For $f(x,y)$, $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ gradient

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f$$

It is convenient to denote

$$\boxed{\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}}$$

Then $\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) \cdot (M\hat{i} + N\hat{j})$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div}\vec{F}$$

Hence we also write

$$\boxed{\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}}$$

Def 13: Define $\operatorname{rot} \vec{F}$ to be

$$\operatorname{rot} \vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad (\text{for } \vec{F} = M\hat{i} + N\hat{j})$$

Note: $\operatorname{rot} \vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\overline{D_\varepsilon}(x,y))} \iint_{\overline{D_\varepsilon}(x,y)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$

$$= \lim_{\varepsilon \to 0} \frac{1}{\operatorname{Area}(\overline{D_\varepsilon}(x,y))} \oint_{\partial \overline{D_\varepsilon}(x,y)} \vec{F} \cdot \hat{T} \, ds$$

$$\overset{\text{(called)}}{=} \text{circulation density}$$

Using $\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y}$, we can write

$$\boxed{\operatorname{rot} \vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}}$$

Since $\vec{F} = M\hat{i} + N\hat{j} + 0 \cdot \hat{k}$ (in $\mathbb{R}^3$) $\begin{pmatrix} M = M(x,y) \\ N = N(x,y) \end{pmatrix}$

$\vec{\nabla} = \hat{i} \frac{\partial}{\partial x} + \hat{j} \frac{\partial}{\partial y} + \hat{k} \frac{\partial}{\partial z}$ (in $\mathbb{R}^3$)

$$\Rightarrow \quad \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix} \hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \hat{k}$$

$\Rightarrow \operatorname{rot} \vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}$, i.e. $\hat{k}$-component of "curl $\vec{F}$" where

$$\boxed{\operatorname{curl}\vec{F} \overset{\text{def}}{=\!=} \vec{\nabla}\times\vec{F}}$$

In these notation, the Green's thm can be written as

Vector form of Green's Thm

normal form
$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \operatorname{div}\vec{F}\,dA}$$

or
$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_D \vec{\nabla}\cdot\vec{F}\,dA}$$

tangential form
$$\boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_D \operatorname{curl}\vec{F}\cdot\hat{k}\,dA}$$

or
$$\boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_D (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA}$$

And Thm 10 can be written as

Thm 10': Ω simply-connected & connected, $\vec{F}\in C^1$.

Then

$\vec{F}$ = conservative $\Leftrightarrow$ $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = 0$

(Check: case for $n=3$)

Note : (i) curl $\vec{F} = \vec{\nabla} \times \vec{F}$ defined only in $\mathbb{R}^3$ ($\supset \mathbb{R}^2$)

(ii) but div $\vec{F} = \vec{\nabla} \cdot \vec{F}$ can be defined in $\mathbb{R}^n$ for any n.

In particular, in $\mathbb{R}^3$

Def 12' The divergence of $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ is defined to be

$$\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(M\hat{i} + N\hat{j} + L\hat{k}\right)$$
$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}$$

Then one can easily check the following facts : (Ex!)

For C^2 function f and C^2 vector field $\vec{F}$:

(i) $\vec{\nabla} \times (\vec{\nabla} f) = 0$ (i.e. curl $\vec{\nabla} f = 0$)

(ii) $\vec{F}$ conservative $\Rightarrow$ curl $\vec{F} = \vec{\nabla} \times \vec{F} = 0$

(iii) $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0$ (i.e. div(curl $\vec{F}$) $= 0$)

? $\vec{\nabla} \cdot (\vec{\nabla} f) =$?