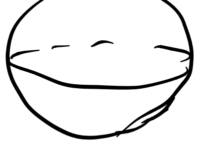
Def 15 A subset $D \subseteq \mathbb{R}^n$, $n=2$ or 3, is called <u>simply-connected</u> if every closed curve in D can be <u>contracted</u> to a point in D without ever leaving D.

(contracted : deformed continuously)

eg 44: Ω_1 in eg 43 is simply-connected, but Ω_2 is <u>not</u> simply-connected.

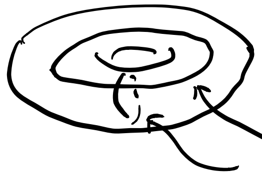
eg 45:



$S^2 \subset \mathbb{R}^3$ $\quad S^2 : x^2+y^2+z^2=1$

is simply-connected

eg 46:



torus $\mathbb{T}^2 \cong S^1 \times S^1 \subset \mathbb{R}^3$ is <u>not</u> simply-connected.

these 2 closed curves cannot be contracted to a point on $\mathbb{T}^2$

Remark: <u>Simply-connectedness</u> is a <u>global</u> condition to guarantee "eqts in Cor to Thm 9" $\Rightarrow$ "conservative".

Thm 10: Suppose $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, is <u>connected</u> and <u>simply-connected</u>. Let $\vec{F}$ be C^1 vector field on Ω. Then

$\vec{F}$ is conservative on $\Omega \Leftrightarrow$ components of $\vec{F}$ satisfy the system of PDE in the cor. to the Thm 9.

(Pf: later)

eg 47 Let $\Omega \equiv \mathbb{R}^3$ (connected & simply-connected)

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$= (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}$$

Find the potential function f of $\vec{F}$, i.e.

$$\vec{\nabla} f = \vec{F}.$$

Soln: This is, we want to solve

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = L.$$

Checking M, N, L satisfy the system of PDE in Cor to Thm 9:

$$\begin{array}{ccc} \frac{\partial M}{\partial x} = 0 & \frac{\partial M}{\partial y} = 1 & \frac{\partial M}{\partial z} = e^z \\ \frac{\partial N}{\partial x} = 1 & \frac{\partial N}{\partial y} = 0 & \frac{\partial N}{\partial z} = 0 \\ \frac{\partial L}{\partial x} = e^z & \frac{\partial L}{\partial y} = 0 & \frac{\partial L}{\partial z} = xe^z \end{array}$$

Thm 10 $\Rightarrow$ existence of potential function f.

To find f explicitly:

$$\frac{\partial f}{\partial x} = y + e^z$$

$$\Rightarrow \quad f = \int (y+e^z)\,dx = x(y+e^z) + \text{"const. in } x\text{"}$$

(function of y & z only)

$$= xy + xe^z + g(y,z) \quad \text{for some function } g(y,z)$$

$$x+1 = \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 1$$

$$\Rightarrow \quad g = y + \text{"const in } y\text{"}$$

$$= y + h(z) \quad \text{for some function } h(z)$$

$$\Rightarrow \quad f = xy + xe^z + y + h(z)$$

$$1 + xe^z = \frac{\partial f}{\partial z} = xe^z + h'(z)$$

$$\Rightarrow \quad h'(z) = 1$$

$$\Rightarrow \quad h = z + \text{const.}$$

Hence $f(x,y,z) = xy + xe^z + y + z + c$, where c is a constant

is the required potential function.

(Note: This is equivalent to find f s.t.
the total differential $df = Mdx + Ndy + Ldz$)

<u>Remark</u>: To prove Thm 10 in $\mathbb{R}^2$, we need the Green's Thm)
(in $\mathbb{R}^3$, we need the Stokes' Thm)

<u>Thm 11</u> (<u>Green's Theorem</u>)

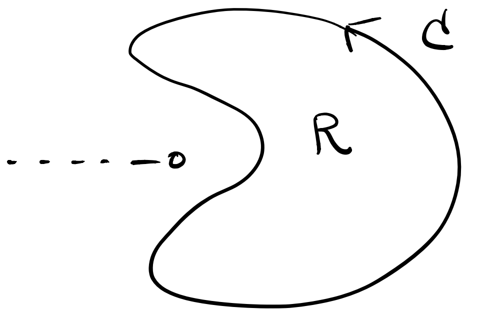
Let $\Omega \subseteq \mathbb{R}^2$ be open, $\vec{F} = M\hat{i} + N\hat{j}$ be C^1 vector field on Ω;

C is a piecewise <u>"smooth"</u> <u>simple closed</u> <u>anti-clockwise oriented</u> curve enclosing a region R which lies entirely in Ω.

Then
- <u>Normal</u> Form $\oint_C \vec{F} \cdot \hat{n}\, ds = \boxed{\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx dy}$
- <u>Tangential</u> Form $\oint_C \vec{F} \cdot \hat{T}\, ds = \boxed{\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy}$

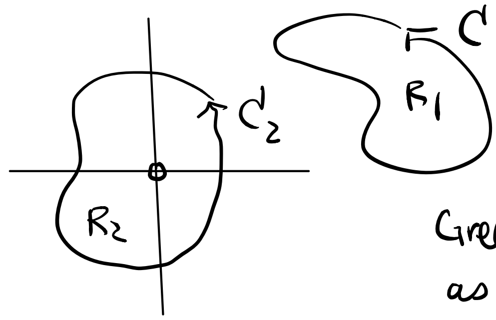
(Remark: The two forms are equivalent)

Note: $\Omega_1 = \mathbb{R}^2 \setminus \{x \le 0\}$



Green's Thm applies as $R \subset \Omega_1$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$



Green's Thm applies as $R_1 \subset \Omega_2$

$R_2 \not\subset \Omega_2$
(since $(0,0) \in R_2$ but $(0,0) \notin \Omega_2$)
Green's Thm doesn't apply.

<u>eg48</u>: Verify both form of Green's Thm for

$$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j} \quad \text{on } \Omega = \mathbb{R}^2 \quad (\text{is } C^\infty)$$

C = unit circle : $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $t \in [0, 2\pi]$.

Then R = region enclosed by $C = \{x^2+y^2<1\}$ the unit disc.
(We also write $C = \partial R$ boundary of R)

<u>Soln</u>:
$$M = x-y, \quad N = x$$
$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0$$

On C, $x = \cos t$, $y = \sin t$ $\quad t \in [0, 2\pi]$

<u>Normal form</u>:
$$\text{LHS} = \oint_C M\,dy - N\,dx$$
$$= \int_0^{2\pi} [(\cos t - \sin t)\cos t - \cos t(-\sin t)]\,dt$$
$$= \pi \qquad (\text{check!})$$

$$\text{RHS} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1+0)\,dA = \pi$$

Tangential form :

$$\text{LHS} = \oint_C M\,dx + N\,dy = 2\pi \quad (\text{check!})$$

$$\text{RHS} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_R (1-(-1))\,dA = 2\pi$$

(Note : This example shows that even the 2 form are equivalent, but values involved may differ.)