Conservative Vector Field

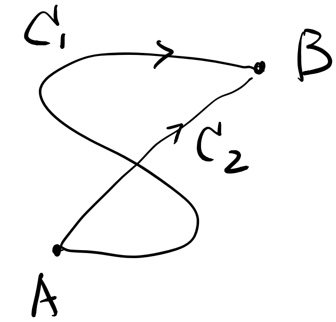
Def 14 Let $\Omega \subset \mathbb{R}^n$, $n=2$ or 3, be open. A vector field $\vec{F}$ defined on Ω is said to be <u>conservative</u> if

$$\int_C \vec{F}\cdot\hat{T}\,ds \left(= \int_C \vec{F}\cdot d\vec{r}\right)$$ along an <u>oriented</u> curve C in Ω

<u>depends</u> only on the <u>starting</u> point and <u>end</u> point of C.

Note: This is usually referred as "<u>path independent</u>". i.e. If C_1 & C_2 are <u>oriented</u> curves with same <u>starting</u> point A and <u>end</u> point B, then

$$\int_{C_1} \vec{F}\cdot\hat{T}\,ds = \int_{C_2} \vec{F}\cdot\hat{T}\,ds$$



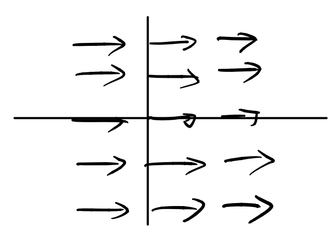
(So the value only depends on the points A & B (& direction))

<u>Notation</u>: If $\vec{F}$ is conservative, we sometimes write

$\int_A^B \vec{F}\cdot\hat{T}\,ds$ to denote the <u>common</u> <u>value</u> of $\int_C \vec{F}\cdot\hat{T}\,ds$ along any oriented curve C from A to B.

eg 41: $\vec{F} \equiv \hat{i}$ on $\mathbb{R}^2$

C: $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $a \le t \le b$



Then $$\int_C \vec{F}\cdot\hat{T}\,dS = \int_C \vec{F}\cdot d\vec{r} = \int_a^b x'(t)\,dt = x(b) - x(a)$$

(x coordinate at $\vec{r}(b)$ and at $\vec{r}(a)$)

$\therefore \int_C \vec{F}\cdot\hat{T}\,ds$ depends only on the starting point & end point.

$\Rightarrow$ $\vec{F}$ is conservative.

(Note: $\vec{F} = \vec{\nabla} f$ where $f(x,y) = x$)

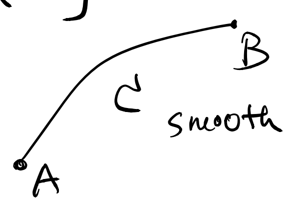
<u>Thm 8</u> (<u>Fundamental Theorem of Line Integral</u>)

Let f be a C^1 function on an open set $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, and $\vec{F} = \vec{\nabla} f$ be the gradient vector field of f. Then for any piecewise smooth oriented curve C on Ω with starting point A and end point B,

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = f(B) - f(A)}$$

Pf: Assume C is a smooth curve parametrized by

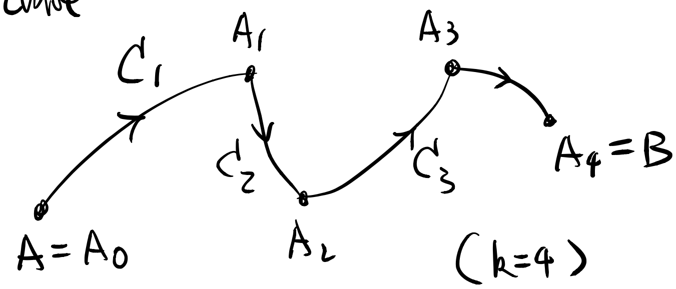
$\vec{r}(t)$, $a \le t \le b$



Then
$$\begin{aligned}
\int_C \vec{F}\cdot\hat{T}\,ds &= \int_C \vec{F}\cdot d\vec{r} \\
&= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt \\
&= \int_a^b \vec{\nabla} f(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt \\
&= \int_a^b \frac{d}{dt} f(\vec{r}(t))\,dt \\
&= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{(by 1-variable fundamental Thm of Calculus)} \\
&= f(B) - f(A)
\end{aligned}$$

For a general piecewise smooth curve

$C = C_1 \cup C_2 \cup \cdots \cup C_k$



$\Big(= C_1 + C_2 + \cdots + C_k$
in order to indicate
that the orientation of C_i are correct wrt the orientation of C $\Big)$

where C_i is smooth going from A_{i-1} to A_i.

Then $\displaystyle\int_C \vec{F} \cdot \hat{T}\, ds = \sum_i \int_{C_i} \vec{F} \cdot \hat{T}\, ds$

$$= \sum_i \left[f(A_i) - f(A_{i-1}) \right]$$

$$= f(A_k) - f(A_0)$$

$$= f(B) - f(A)$$

※

Thm 9 Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open and connected
$\vec{F}$ is a continuous vector field on Ω. Then the following are equivalent.

(a) $\exists$ a C^1 function $f : \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f.$$

(b) $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = 0$ along any closed curve C on Ω.

(c) $\vec{F}$ is conservative.

Pf: "(a) ⇒ (b)" If f is C^1 and $\vec{F} = \vec{\nabla} f$

and $\vec{r}: [a,b] \to \Omega$ parametrizes C (any closed curve)

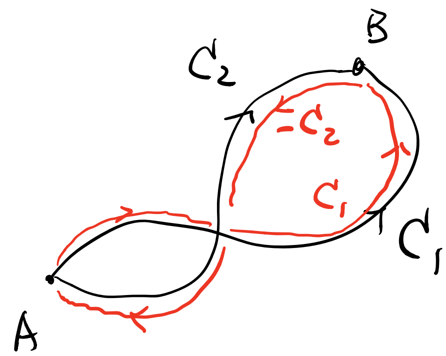
C closed $\Rightarrow \vec{r}(a) = \vec{r}(b) = A$

Fundamental Thm of Line Integral ⇒

$$\oint_C \vec{F} \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)) = f(A) - f(A) = 0.$$

"(b) ⇒ (c)" Suppose C_1, C_2 are oriented curves with

starting A and end point B



Then $C_1 \cup (-C_2)$

$= C_1 - C_2$ (better notation)

is an oriented closed curve.

Then by (b)

$$0 = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{-C_2} \vec{F} \cdot d\vec{r}$$

$$= \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r}$$

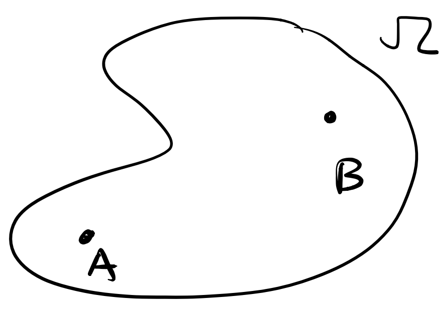
Since C_1 & C_2 are arbitrary, $\vec{F}$ is conservative.

"(c) ⇒ (a)" Assume $n = 2$ for simplicity (other dimensions are similar)

Let $\vec{F} = M\hat{i} + N\hat{j}$ are conservative

Fix a point $A \in \Omega$

For any point $B \in \Omega$,



$$f(B) = \int_A^B \vec{F}\cdot\hat{T}\,ds = \text{common value of } \int_C \vec{F}\cdot\hat{T}\,ds$$
for any C from A to B.
(since $\vec{F}$ is conservative)

[We've also used the assumption that Ω is connected, otherwise there is no path from A to B if A, B belong to different connected components:

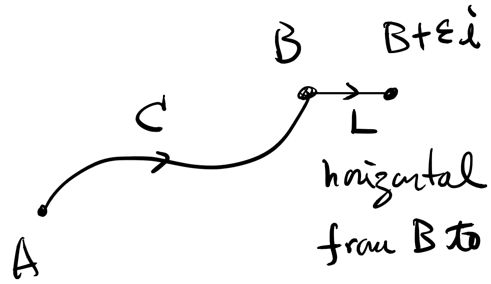
]

Hence $f(B)$ is well-defined

Claim $\vec{F} = \vec{\nabla} f$

Pf of claim: $\dfrac{\partial f}{\partial x}(B) = \lim\limits_{\varepsilon \to 0} \dfrac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon}$

Let C be an oriented curve from A to B.



horizontal line segment from B to $B+\varepsilon\hat{i}$ with $|\varepsilon|$ sufficiently small (s.t. $C+L \subset \Omega$) Ω open

Then $f(B+\varepsilon\hat{i})$

$$= \int_A^{B+\varepsilon\hat{i}} \vec{F}\cdot d\vec{r}$$

$$= \int_{C+L} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_L \vec{F}\cdot d\vec{r}$$

$$= \int_A^B \vec{F}\cdot d\vec{r} + \int_L \vec{F}\cdot d\vec{r} = f(B) + \int_L \vec{F}\cdot d\vec{r}$$

$$\therefore \quad \frac{f(B+\varepsilon\hat{i})-f(B)}{\varepsilon} = \frac{1}{\varepsilon}\int_L \vec{F}\cdot d\vec{r}$$

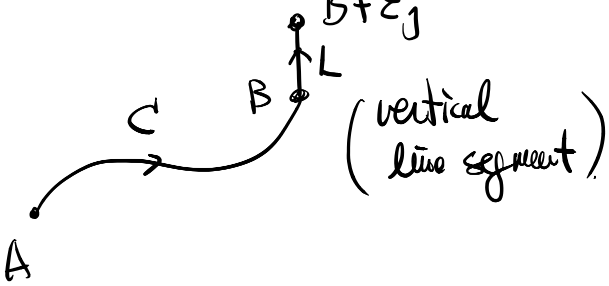
$$= \frac{1}{\varepsilon}\int_0^{\varepsilon} M(x+t,y)\,dt$$

(parametrise L by $B+t\hat{i}$, $t\in[0,\varepsilon]$, $B=(x,y)$)

$$\therefore \quad \frac{\partial f}{\partial x}(B) = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^{\varepsilon} M(x+t,y)\,dt$$

$$= M(x,y) \quad (\text{by MVT \& } M \text{ is continuous})$$

Similarly $\frac{\partial f}{\partial y}(B) = N(x,y)$ by considering



So $\vec{\nabla} f = \vec{F}$.

Since $\vec{F}$ is continuous,

i.e. $M = \frac{\partial f}{\partial x}$ & $N = \frac{\partial f}{\partial y}$ are continuous

$\Rightarrow$ f is C^1 ※

Remark: The function f in (a) of Thm 9 is called the potential function for $\vec{F}$. It is unique up to an additive constant: $\vec{\nabla}(f+c) = \vec{F}$, $\forall$ const. c.

Corollary (to Thm 9)

Let $\vec{F}$ be conservative and C^1

connected

"n=3" If $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ (on $\Omega \subset \mathbb{R}^3$)

then
$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y} \\ \dfrac{\partial L}{\partial x} = \dfrac{\partial M}{\partial z} \end{cases}$$

connected

"n=2" If $\vec{F} = M\hat{i} + N\hat{j}$ (on $\Omega \subset \mathbb{R}^2$)

then $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Pf: $\vec{F}$ conservative $\xrightarrow{\text{Thm } 9}$ $\vec{F} = \vec{\nabla} f$ for some function f

i.e. $$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$
$$= M\hat{i} + N\hat{j} + L\hat{k} = \vec{F}$$

$\vec{F} \in C^1 \Rightarrow f \in C^2$. Hence mixed derivatives thm (Clairaut's Thm)

$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial N}{\partial x} \\ \dfrac{\partial N}{\partial z} = \dfrac{\partial^2 f}{\partial z \partial y} = \dfrac{\partial^2 f}{\partial y \partial z} = \dfrac{\partial L}{\partial y} \\ \dfrac{\partial L}{\partial x} = \dfrac{\partial^2 f}{\partial x \partial z} = \dfrac{\partial^2 f}{\partial z \partial x} = \dfrac{\partial M}{\partial z} \end{cases}$$

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(Similarly for n=2)

eg 42 Show that $\vec{F}(x,y) = \hat{i} + x\hat{j}$ is <u>not</u> conservative in $\mathbb{R}^2$.

Sol^n $(\vec{F} \in C^\infty)$ $\begin{cases} M \equiv 1 \\ N = x \end{cases} \Rightarrow \dfrac{\partial M}{\partial y} = 0 \neq 1 = \dfrac{\partial N}{\partial x}$

By Cor to Thm 9, $\vec{F}$ is not conservative. #

<u>Remark (Important)</u>

For a C^1 vector field $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$\boxed{\vec{F} \text{ conservative}}$ $\underset{?}{\overset{\text{Cor to Thm 9}}{\rightleftarrows}}$ $\boxed{M, N, L \text{ satisfy the system of PDE in the Cor to Thm 9}}$

Answer: <u>NOT</u> true in general, needs extra condition on the domain Ω. ("connected" is not enough)

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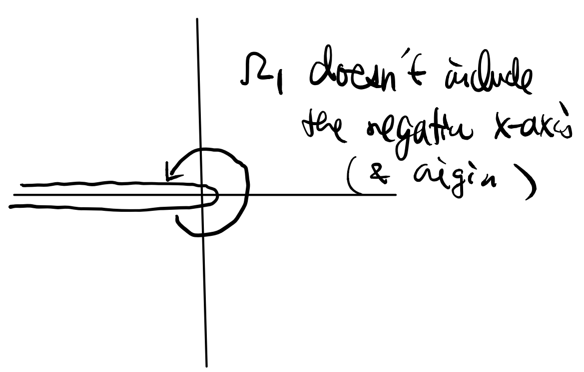
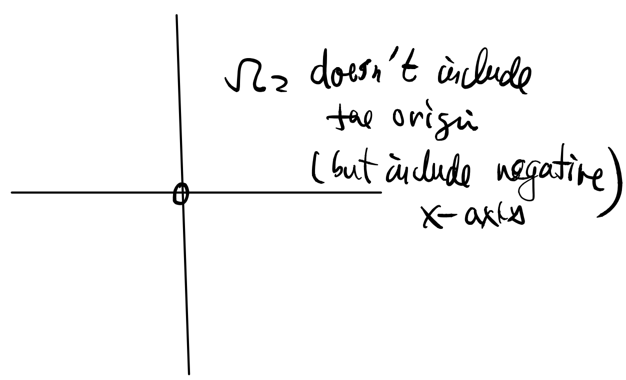
eg 43 Consider the vector field

$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

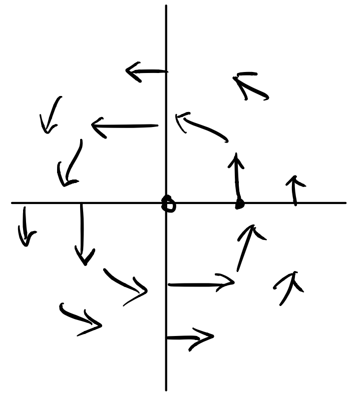
and the domains
$$\Omega_1 = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$$
$$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$$





In polar cordinates

$$\vec{F} = -\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}$$



$\Rightarrow$ $\vec{F}$ rotates around the origin anti-clockwisely

$|\vec{F}| = \frac{1}{r} \to 0$ as $r \to +\infty$

$|\vec{F}| = \frac{1}{r} \to +\infty$ as $r \to 0$ $\Rightarrow$ $\vec{F}$ cannot be extended to a C^1 vectorfield on $\mathbb{R}^2$.

Besides $(0,0)$, $\vec{F}$ is C^1 and hence $\vec{F}$ is C^1 in Ω_1 and also C^1 in Ω_2.

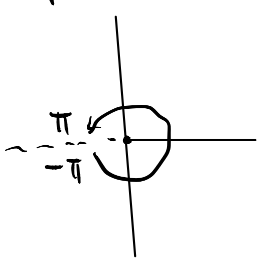
Questions: Is $\vec{F}$ conservative on Ω_1?
Is $\vec{F}$ conservative on Ω_2?

<u>Soln</u>: (1) For Ω_1 and (x,y) can be expressed in polar cordinates with

$$\begin{cases} r > 0 \\ -\pi < \theta < \pi \end{cases}$$

((r,θ) are unique)

Define $f(x,y) = \theta$ "smooth" on Ω_1 (Check!)



Then

$$\begin{cases} \frac{\partial f}{\partial x} = -\frac{\sin\theta}{r} \\ \frac{\partial f}{\partial y} = \frac{\cos\theta}{r} \end{cases}$$

$$\left(\theta_x = -\frac{\sin\theta}{r}, \ \theta_y = \frac{\cos\theta}{r} \right)$$

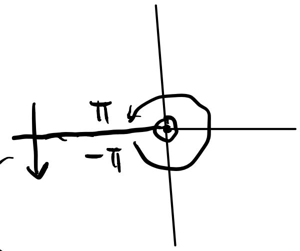
$\Rightarrow$ $\vec{F} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \vec{\nabla} f$

$\Rightarrow$ $\vec{F}$ is conservative.

(2) For Ω_2, the function

$f(x,y) = \theta$ cannot be extended to a "smooth" function on (the whole) Ω_2

$\therefore$ $f(x,y) = \theta$ doesn't work in the case of Ω_2.



(the function $f = \theta$ "jump" at the negative x-axis
$\Rightarrow$ f cannot be extended to continuous function across $-$x-axis)

To show that $\vec{F}$ is <u>not</u> conservative, we consider a <u>closed curve</u>

$$C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [-\pi, \pi]$$

(unit circle in Ω_2, but it is not a curve in Ω_1)

Then
$$\oint_C \vec{F} \cdot d\vec{r} = \int_{-\pi}^{\pi} \left(-\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j}\right) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_{-\pi}^{\pi} \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right) \cdot \left(-\sin t\,\hat{i} + \cos t\,\hat{j}\right) dt$$

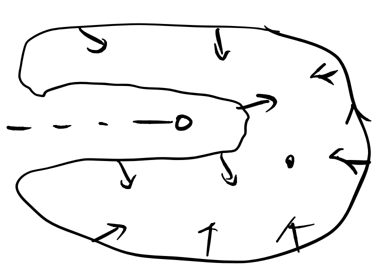
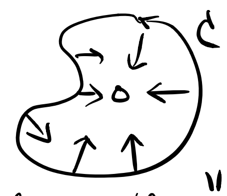
($r = 1$, & $\theta = t$ along C)

$$= \int_{-\pi}^{\pi} 1\,dt$$

$$= 2\pi \neq 0$$

By Thm 9, $\vec{F}$ is not conservative. $\#$

Summary

Ω_1	Ω_2
$f(x,y) = \theta$ smooth function on Ω_1	$f(x,y) = \theta$ is **not** a smooth function on Ω_2 (θ cannot be well-defined on the whole Ω_2)
$C : x^2+y^2=1$ is **not** a curve in Ω_1 because $(-1,0) \in C$ but $(-1,0) \notin \Omega_1$	$C : x^2+y^2=1$ is a closed curve in Ω_2
 Closed curve **cannot** circle around the origin $\Rightarrow$ closed curves can be deformed continuous (within Ω_1) to a point (in Ω_1)	 C enclosed the "hole" $\Rightarrow$ C cannot be deformed continuous (within Ω_2) to a point (in Ω_2)