Another way to justify the notation:

$$\vec{r} = (x, y, z) \quad \text{the position vector}$$

$$\Rightarrow \quad \boxed{d\vec{r} = (dx, dy, dz)} \quad \text{(naturally)}$$

Then $$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_C (M,N,L)\cdot(dx,dy,dz)$$

$$= \int_C M\,dx + N\,dy + L\,dz.$$

eg39: Evaluate $I = \int_C -y\,dx + z\,dy + 2x\,dz$

where $C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \quad (0 \le t \le 2\pi)$

$$= (\cos t, \sin t, t)$$

Soln: $d\vec{r} = (-\sin t, \cos t, 1)\,dt$

$$\Rightarrow \quad I = \int_0^{2\pi} [-\sin t(-\sin t) + t\cos t + 2\cos t(-1)]\,dt$$

$$= \cdots = \pi$$

Physics

(1) $\vec{F}$ = Force field

C = oriented curve

then $$\boxed{W = \int_C \vec{F}\cdot\hat{T}\,ds}$$

is the work done in moving an object along C.

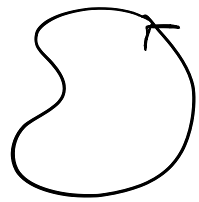
(2) $\vec{F}$ = velocity vector field of fluid

C = oriented curve

then
$$\boxed{\text{Flow} = \int_C \vec{F} \cdot \hat{T}\, ds}$$

Flow along the curve C.

If C is closed curve, the flow is also called a circulation



C

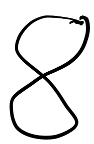
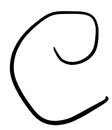
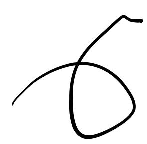
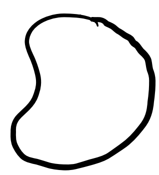
Def 13: A curve is said to be

(i) simple if it does not intersect with itself except possibly at end points.

(ii) closed if starting point = end point.
(also called a loop)

(iii) simple closed curve if it is both simple and closed.

Note:

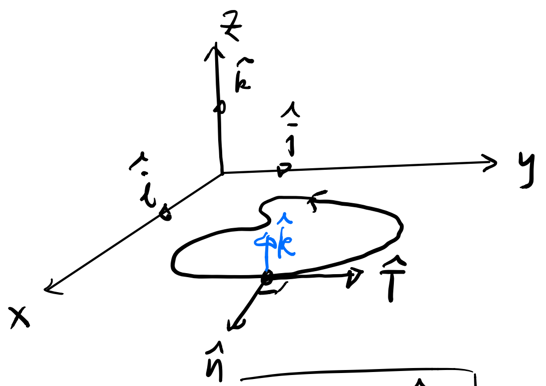
				
simple	NO	Yes	NO	Yes
closed	Yes	NO	NO	Yes

(3) $\vec{F}$ = velocity of fluid

C = oriented plane curve ($C \subset \mathbb{R}^2$)

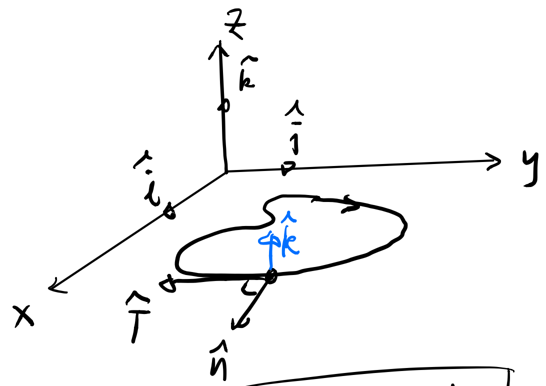
parametrized by $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$

$\hat{n}$ = outward-pointing unit normal to the curve C



$$\boxed{\hat{n} = \hat{T} \times \hat{k}}$$

if C is of anti-clockwise orientation



$$\boxed{\hat{n} = -\hat{T} \times \hat{k}}$$

if C is of clockwise orientation

Formula for $\hat{n}$ (wrt the parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$)

Recall $$\hat{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{x'(t)\hat{i} + y'(t)\hat{j}}{|\vec{r}\,'(t)|}$$

(in arc-length parameter: $\hat{T} = \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}$)

Anti-clockwise:

$$\hat{n} = \hat{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{x'}{|\vec{r}\,'|} & \frac{y'}{|\vec{r}\,'|} & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{y'(t)\hat{i} - x'(t)\hat{j}}{|\vec{r}\,'(t)|}$$

(or $\hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$)

Clockwise : $\hat{n} = \dfrac{-y'(t)\hat{i} + x'(t)\hat{j}}{|\vec{r}\,'(t)|}$

$\left(\text{or} \quad \hat{n} = -\dfrac{dy}{ds}\hat{i} + \dfrac{dx}{ds}\hat{j} \right)$

$$\boxed{\text{Flux of } \vec{F} \text{ across } C \overset{\text{def}}{=\!=} \int_C \vec{F} \cdot \hat{n}\, ds}$$

(amount of fluid getting out of the closed curve C)

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$
and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is anti-clockwise parametrization of C (closed curve)

Then

$$\boxed{\begin{aligned} &\text{Flux of } \vec{F} \text{ across } C \\ &= \oint_C (M\hat{i} + N\hat{j}) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \right) ds \\ &= \oint_C M\,dy - N\,dx \end{aligned}}$$

Remark :
- $\ointctrclockwise$: curve is closed & in anti-clockwise direction
- $\ointclockwise$: curve is closed & in clockwise direction
- But in some books, only "$\oint$" is used, NO arrow, Then one needs to determine the orientation from the context.
- Convention : If no orientation is mentioned, "$\oint$" without arrow means anti-clockwise orientation (positive orientation)

eg 40 : Let $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: \ x^2+y^2=1$

Find the flow (anti-clockwisely) along C and flux across C.

Soln : Let $\vec{F}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

(note : correct orientation)

Then

$$\text{flow} = \oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^{2\pi} [(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}]\cdot[-\sin t\,\hat{i} + \cos t\,\hat{j}]\,dt$$

$$= \cdots = 2\pi \quad (\text{Check!})$$

$$\text{flux} = \oint_C \vec{F}\cdot\hat{n}\,ds$$

$$= \oint_C M\,dy - N\,dx \quad (\text{with anti-clockwise parametrization})$$

$$= \int_0^{2\pi} (\cos t - \sin t)\,d\sin t - \cos t\,d\cos t$$

$$= \int_0^{2\pi} [(\cos t - \sin t)\cos t + \cos t \sin t]\,dt$$

$$= \cdots = \pi \quad (\text{check!})$$

#

Remark : If C is an oriented curve, then denote by "$-C$" the oriented curve with opposite orientation

- If f is a scalar function

$$\boxed{\int_C f\,ds = \int_{-C} f\,ds}$$

as "ds" is not oriented, just "length"

- If $\vec{F}$ is a vector field

flow

$$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = -\int_{-C} \vec{F}\cdot\hat{T}\,ds}$$

this $\hat{T}$ is the "$\hat{T}$ for $-C$"

More precisely, we should write

$$\int_C \vec{F}\cdot\hat{T}_C\,ds = -\int_{-C} \vec{F}\cdot\hat{T}_{-C}\,ds$$

- But for flux

$$\boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \oint_{-C} \vec{F}\cdot\hat{n}\,ds}$$

Summary

scalar f	$\int_C f\,ds$ indep. of orientation	f, ds have no direction
vector $\vec{F}$ flow	$\int_C \vec{F}\cdot\hat{T}\,ds$ depends on orientation	$\hat{T}$ depends on direction
flux	$\int_C \vec{F}\cdot\hat{n}\,ds$ indep. of orientation	$\hat{n}$ = always outward