(cont'd from previous notes)

$$J(H) = \begin{pmatrix} 1 & 0 \\ \frac{\partial h}{\partial y_1} & \frac{\partial h}{\partial y_2} \end{pmatrix}$$

$$\det J(H) = \frac{\partial h}{\partial y_2} = \frac{\partial f_2}{\partial x_1}\frac{\partial x_1}{\partial y_2} + \frac{\partial f_2}{\partial x_2}\frac{\partial x_2}{\partial y_2} \quad \text{(Chain rule)}$$

$$= \frac{\partial f_2}{\partial x_1} \cdot \frac{\partial g}{\partial y_2} + \frac{\partial f_2}{\partial x_2} \cdot 1$$

$$= \frac{\partial f_2}{\partial x_1}\left(-\frac{\partial f_1}{\partial x_2}\frac{\partial g}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2}$$

$$= -\frac{\frac{\partial f_2}{\partial x_1}\frac{\partial f_1}{\partial x_2}}{\frac{\partial f_1}{\partial x_1}} + \frac{\partial f_2}{\partial x_2}$$

$$= \frac{1}{\frac{\partial f_1}{\partial x_1}} \cdot \left[\frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1}\right]$$

$$= \frac{\det J(F)}{\frac{\partial f_1}{\partial x_1}} \neq 0 \quad \text{at } p.$$

So, H & K satisfy the requirements and we have

$$H \circ K \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = H \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_1 \\ h(y_1, y_2) \end{pmatrix}$$

$$= \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This completes case 1.

<u>Case 2</u> $\frac{\partial f_1}{\partial x_1}(p) = 0$

Since $\det J(F) = \frac{\partial f_1}{\partial x_1}\frac{\partial f_2}{\partial x_2} - \frac{\partial f_1}{\partial x_2}\frac{\partial f_2}{\partial x_1} \neq 0$ at p,

$\frac{\partial f_1}{\partial x_2}(p) \neq 0$

Interchanging the variables $\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$

Then the new mapping $\tilde{F}\begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix}$ satisfies the condition in case 1. Applying case 1 to $\tilde{F}$, then interchanging back to x_1, x_2 ※

<u>Step 2</u> Let $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = K\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} k(x_1, x_2) \\ x_2 \end{pmatrix}$ be a diffeomorphism from region R_1 to $R_2 = K(R_1)$. Then for any function $f(y_1, y_2)$ on R_2,

$$\iint_{R_2} f(y_1, y_2)\, dy_1 dy_2 = \iint_{R_1} f \circ K(x_1, x_2) \left|\det J(K)\right| dx_1 dx_2$$

$$= \iint_{R_1} f(k(x_1, x_2), x_2) \left|\frac{\partial(y_1, y_2)}{\partial(x_1, x_2)}\right| dx_1 dx_2$$

<u>Pf</u>: By additivity property of integrations and cutting R_1 (and correspondingly $R_2 = K(R_1)$) into small regions, we may assume $R_1 = [a,b] \times [c,d] = \{a \leq x_1 \leq b,\ c \leq x_2 \leq d\}$

For any fixed $y_2 = x_2$,

$$y_1 = k(x_1, x_2) = k(x_1, y_2), \text{ for } a \leq x_1 \leq b$$

can be regarded as a transformation of 1-variable

Note that $\frac{\partial y_1}{\partial x_1} = \frac{\partial k}{\partial x_1} = \det \begin{pmatrix} \frac{\partial k}{\partial x_1} & \frac{\partial k}{\partial x_2} \\ 0 & 1 \end{pmatrix} = \det J(K) \neq 0$.

(Step 1)

Note also that R_2 is of special form

$$\{c \le y_2 \le d,\ k(a,y_2) \le y_1 \le k(b,y_2)\} \quad (\text{if } \tfrac{\partial y_1}{\partial x_1} > 0)$$

or $$\{c \le y_2 \le d,\ k(b,y_2) \le y_1 \le k(a,y_2)\} \quad (\text{if } \tfrac{\partial y_1}{\partial x_1} < 0)$$

By Fubini's Thm (assuming $\frac{\partial y_1}{\partial x_1} > 0$, the other case is similar)

$$\iint_{R_2} f(y_1,y_2)\,dy_1\,dy_2 = \int_c^d \left(\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\,dy_1 \right) dy_2$$

and change of variable formula in 1-variable implies

$$\int_{k(a,y_2)}^{k(b,y_2)} f(y_1,y_2)\,dy_1 = \int_a^b f(k(x_1,y_2),y_2)\,\frac{\partial y_1}{\partial x_1}\,dx_1$$

$$= \int_a^b f(k(x_1,x_2)),x_2)\,\frac{\partial y_1}{\partial x_1}\,dx_1 \quad \left(\text{since } x_2 = y_2 \text{ is fixed}\right)$$

$$\Rightarrow \iint_{R_2} f(y_1,y_2)\,dy_1\,dy_2 = \int_c^d \left(\int_a^b f(k(x_1,x_2),x_2)\,\frac{\partial y_1}{\partial x_1}\,dx_1 \right) dx_2$$

$$= \int_c^d \int_a^b f(k(x_1,x_2),x_2)\,|\det J(K)|\,dx_1\,dx_2$$

$\left(\text{since } \frac{\partial y_1}{\partial x_1} > 0\right)$

$$= \iint_{R_1} f(k(x_1,x_2),x_2)\,|\det J(K)|\,dx_1\,dx_2 \quad \#$$

Step 3 : If the change of variables formula holds for F & G, then it holds for $F \circ G$

Pf: Easily by $J(F \circ G) = J(F) J(G)$ (Chain Rule)

$$\Rightarrow \quad |\det J(F \circ G)| = |\det J(F)| \, |\det J(G)|$$

#

Final step: Combining steps 1-3, and using additivity property of integration, we've proved the Thm 6 for general change of variables formula ##

[Actually, this applies to all dimensions.]

Vector Analysis

Notation: Usually in textbooks, vectors are denoted by boldface **i**, but hard to do it on screen.

so my notation of vectors are:

$$\begin{cases} \text{general vectors: } \vec{v}, \vec{F}, \vec{r}, \vec{\nabla}, \dots \quad (\text{differential operator}) \\ \text{unit vectors: } \hat{i}, \hat{j}, \hat{k}, \hat{n}, \dots \end{cases}$$

Line integrals in $\mathbb{R}^3$ ($\mathbb{R}^n$)

(path integrals)

Ref 9 The line integral of a function f on a curve (path, line) C with parametrization

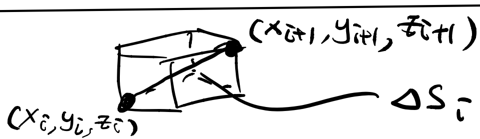
$$\begin{aligned} \vec{r} : [a,b] &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto \vec{r}(t) = (x(t), y(t), z(t)) \end{aligned}$$

is $\displaystyle\int_C f(\vec{r})\,ds = \lim_{\|P\|\to 0} \sum_{i=1}^{n} f(\vec{r}(t_i))\,\Delta s_i$

where P is a partition of $[a,b]$, and

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

(i.e. ds = length element of the curve)



Remarks:

(1) If $f \equiv 1$, $\boxed{\int_C ds = \text{arc-length of } C}$

(2) The definition is well-defined, i.e. the RHS in the definition is independent of the parametrization $\vec{r}(t)$.

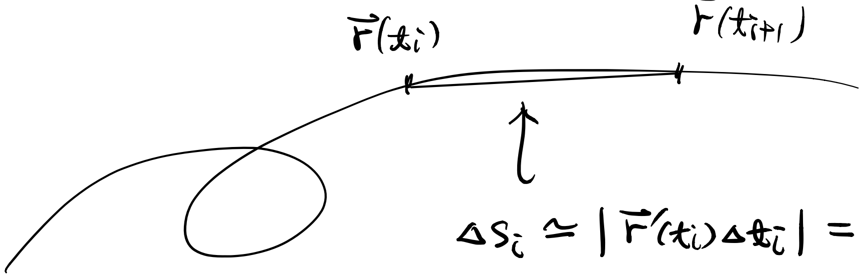
Def 9' (Formula for line integral)

Notations as in Def 9, then

$$\int_C f(\vec{r})\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

where $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$.

Since



$\Delta s_i \simeq |\vec{r}\,'(t_i)\Delta t_i| = |\vec{r}\,'(t_i)|\,\Delta t_i$

(if the curve is differentiable)

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2}\,\Delta t_i$$

$$\cong \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\,\Delta t_i$$

$$= |\vec{r}\,'(t_i)|\,\Delta t_i$$

Remarks (1) "$ds = |\vec{r}\,'(t)|\,dt$" is usually refered as the arc-length element, where $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

and $|\vec{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$.

(2) Suppose the curve C is parametrized by a new parameter $\tilde{t}$

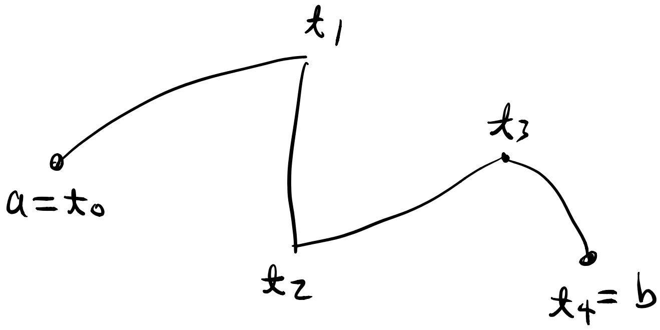
$$\begin{array}{ccc} t & \longleftrightarrow & \tilde{t} \\ \uparrow & & \uparrow \\ [a,b] & & [\tilde{a},\tilde{b}] \end{array} \qquad \left(t \leftrightarrow \tilde{t} \text{ is increasing } \quad \frac{d\tilde{t}}{dt} > 0, \ \frac{dt}{d\tilde{t}} > 0 \right)$$

$$ds = |\vec{r}'(t)|\,dt = \left|\frac{d\vec{r}}{dt}(t)\right| dt$$

$$= \left|\frac{d\vec{r}}{d\tilde{t}} \cdot \frac{d\tilde{t}}{dt}\right| dt = \left|\frac{d\vec{r}}{d\tilde{t}}\right| d\tilde{t} \qquad \text{(by chain rule)}$$

$\therefore$ ds and hence $\int_C f(\vec{r})\,ds$ is <u>independent</u> of the parametrization of C.

(3) If $\vec{r}(t)$ is only piecewise differentiable, then the RHS of <u>Def 9</u> becomes <u>sum</u> of each piece:



If $[a,b] = [t_0, t_1] \cup \cdots \cup [t_{i-1}, t_i] \cup \cdots \cup [t_{k-1}, t_k]$, where $t_0 = a$, $t_k = b$,

such that $\vec{r}\,\Big|_{[t_{i-1}, t_i]}$ is differentiable, then

$$\boxed{\int_C f(\vec{r})\,ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\vec{r}(t))\,|\vec{r}'(t)|\,dt}$$

eg32 $f(x,y,z) = x - 3y^2 + z$

C = line segment joining the origin and $(1,1,1)$

Find $\int_C f(x,y,z)\,ds$.

Soln: Parametrize C by

$$\vec{r}(t) = t(1,1,1) = (t,t,t) \quad , \quad t \in [0,1]$$

(i.e. $x(t) = t$, $y(t) = t$, $z(t) = t$)

$$\Rightarrow \quad \vec{r}'(t) = (1,1,1) \quad , \quad \forall\, t \in [0,1]$$

$$\Rightarrow \quad |\vec{r}'(t)| = \sqrt{3}$$

Hence $\int_C f(x,y,z)\,ds = \int_0^1 f(t,t,t)\sqrt{3}\,dt$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\,dt = 0 \quad \text{(check!)}$$

#

eg33 Let C be curve in $\mathbb{R}^2$ (i.e. $z(t) \equiv 0$)

and it has 2 parametrizations:

$$\vec{r}_1(t) = (\cos t, \sin t) \quad , \quad t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\vec{r}_2(t) = (\sqrt{1-t^2}, -t) \quad , \quad t \in [-1,1]$$

Suppose $f(x,y) = x$. Find $\int_C f(x,y)\,ds$.

(We simply omit the z-variable, as C is a plane curve and f is indep. of z)

Soln: (i) $\vec{r}_1(t) = (\cos t, \sin t)$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$\int_C f(x,y)\,ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(\cos t, \sin t)\,\left|(\cos t, \sin t)'\right|\,dt$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \left|(-\sin t, \cos t)\right|\,dt$$

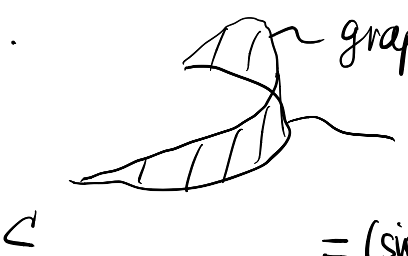
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\,dt = 2 \quad \text{(check!)}$$

(ii) $\vec{r}_2(t) = (\sqrt{1-t^2}, -t)$, $-1 \le t \le 1$

$$\int_C f(x,y)\,ds = \int_{-1}^{1} \sqrt{1-t^2}\,\sqrt{\left(\frac{d}{dt}\sqrt{1-t^2}\right)^2 + \left(\frac{d}{dt}(-t)\right)^2}\,dt$$

$$\underbrace{\cdots}_{\text{(check!)}} = \int_{-1}^{1} dt = 2 \quad \text{(check!)}$$

This verifies the fact that the line integral is indep. of the parametrization.



= (signed) area under the graph of f over C.

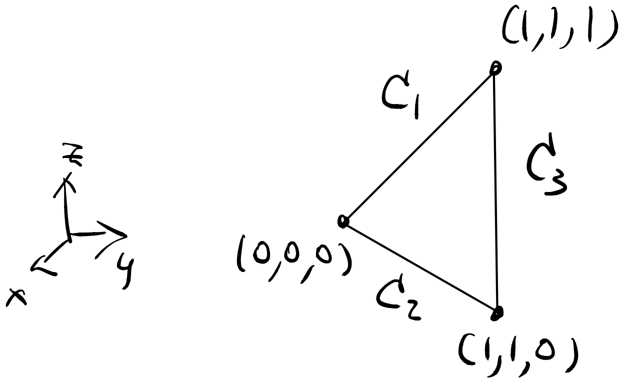
$$\boxed{\begin{array}{l}\underline{\text{Prop 7}}\text{: If } C \text{ is a piecewise smooth curve made by} \\ \text{joining } C_1, C_2, \ldots C_n \text{ end-to-end, then} \\ \displaystyle \int_C f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds\end{array}}$$

(Pf: Clear from the remark of Ref 8)

Remark: "end-to-end" means
"end point of C_{k-1} = initial (end) point of C_k".

eg34: Let $f(x,y,z) = x-3y^2+z$ (again)

C_1, C_2, C_3 are line segments as in the figure:



We already did $\int_{C_1} f\,ds = 0$ (eg32)

One can similarly do $\int_{C_2 \cup C_3} f\,ds = \int_{C_2} f\,ds + \int_{C_3} f\,ds$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2} \quad \text{(ex!)}$$

The observation is $\int_{C_1} f\,ds = 0 \neq \int_{C_2 \cup C_3} f\,ds$

even C_1 & $C_2 \cup C_3$ have the same end points!

Conclusion: Line integral of a function depends, not only on the end points, but also the path.