Def 7 Define the Jacobian $J(u,v)$ of the "coordinates" transformation

$$\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$$

by

$$J(u,v) \underset{\nearrow}{=} \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

notation

With this notation, we should have the formula

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \iint_G f(x(u,v),y(u,v))\,|J(u,v)|\,du\,dv$$

eg 28

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\Rightarrow J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = r \quad \text{(check!)}$$

and

$$\iint_R f(x,y)\,dx\,dy = \iint_G f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

(same formula as before)

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Thm 6: Suppose $\phi: \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomorphism (1-1, onto, ϕ & $\phi^{-1} \in C^1$) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in xy-plane (except possibly on the boundary). Suppose $f(x,y)$ is continuous on R, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

Notes: (i) $f\circ\phi(u,v) = f(x(u,v), y(u,v))$

(ii) ϕ is a diffeomorphism $\Rightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \neq 0$.

Triple integrals (substitutions in triple integrals)

$$\phi(u,v,w) = (x,y,z) : G^{\subset \mathbb{R}^3} \to D^{\subset \mathbb{R}^3}$$

with

$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$$

1-1, onto, cont. differentiable and inverse also cont. differentiable.

Def 8 Jacobian (determinant) of transformation in $\mathbb{R}^3$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}$$

$$= \det \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{pmatrix}$$

Note: Chain rule $\Rightarrow$

$$\begin{cases} \text{2-dim} & \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(s,t)} = \dfrac{\partial(x,y)}{\partial(s,t)} \\ \text{3-dim} & \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(s,t,r)} = \dfrac{\partial(x,y,z)}{\partial(s,t,r)} \end{cases}$$

(Ex!)

$$\Rightarrow \begin{cases} \text{2-dim} & \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\frac{\partial(x,y)}{\partial(u,v)}} \\ \text{3-dim} & \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{\frac{\partial(x,y,z)}{\partial(u,v,w)}} \end{cases}$$

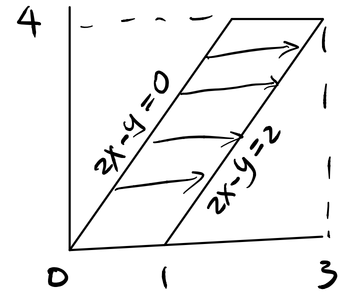
(Ex!)

Thm 7 : Under similar conditions of Thm 6

$$\iiint_D F(x,y,z)\,dxdydz = \iiint_G F(\phi(u,v,w))\,|J(u,v,w)|\,dudvdw$$

$$= \iiint_G F(g(u,v,w), h(u,v,w), k(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dudvdw$$

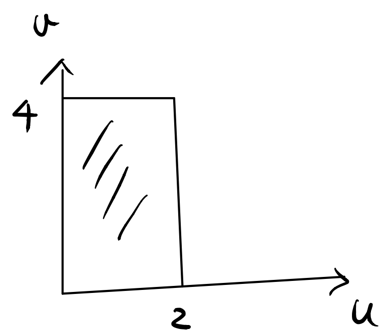
eg 29 $\displaystyle \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dxdy$



lower limit $x=\frac{y}{2} \longleftrightarrow 2x-y=0$

upper limit $x=\frac{y}{2}+1 \longleftrightarrow 2x-y=2$

Define $\begin{cases} u = 2x-y \\ v = y \end{cases}$



Then $\begin{cases} x = \frac{1}{2}u+\frac{1}{2}v \\ y = v \end{cases}$

$\begin{cases} 2x-y=0 \longleftrightarrow u=0 \\ 2x-y=2 \longleftrightarrow u=2 \end{cases}$ $\qquad J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{pmatrix} = \frac{1}{2}$

$\begin{cases} y=0 \\ y=4 \end{cases} \longleftrightarrow \begin{matrix} v=0 \\ v=4 \end{matrix}$

$\therefore \displaystyle \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dxdy = \int_0^4 \int_0^2 \frac{u}{2}\left|\frac{1}{2}\right| dudv = 2$ (check!)

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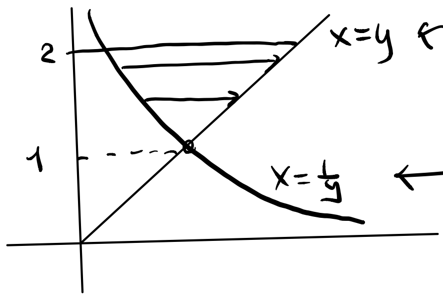
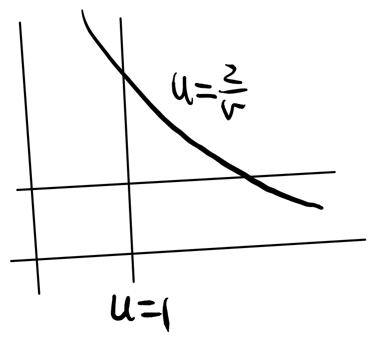
eg 30 $\quad I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\,dy$

Domain of integration



Let $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$ (this should simplify the integration)

Then "boundary" curves $\begin{cases} x = y \leftrightarrow v = 1 \\ x = \frac{1}{y} \leftrightarrow u = 1 \\ y = 2 \leftrightarrow uv = 2 \end{cases}$



And the Jacobian (determinant)

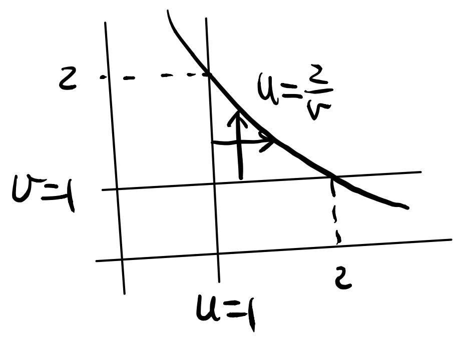
$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$$

Express x, y in terms of u, v 1st :

$uv = y, \quad \frac{u}{v} = x \qquad$ or $\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$

$$\Rightarrow \quad \frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix} = \frac{2u}{v} \quad \text{(check!)}$$

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$



$$= \int_1^2 \int_1^{\frac{2}{v}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv$$

$$\alpha = \int_1^2 \int_1^{\frac{2}{u}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du$$

Let do

$$\int_1^2 \int_1^{\frac{2}{u}} v e^u \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\, du = \int_1^2 \int_1^{\frac{2}{u}} \cancel{v} e^u \frac{2u}{\cancel{v}}\, dv\, du$$

$$= \int_1^2 (2ue^u)\left(\int_1^{\frac{2}{u}} dv\right) du$$

$$= \int_1^2 (4e^u - 2ue^u)\, du \qquad \text{(check!)}$$

$$= 2e(e-2) \qquad \text{(By integration-by-parts)}$$

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<u>eg18 revisit</u> Volume of Ellipsiod.

$$D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1 \right\} \qquad (a, b, c > 0)$$

$$\mathrm{Vol}(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

Change of variables

$$\begin{cases} u = \frac{x}{a} \\ v = \frac{y}{b} \\ w = \frac{z}{c} \end{cases} \iff \begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{pmatrix} a & & \\ & b & \\ & & c \end{pmatrix} = abc \ (>0)$$

$$\mathrm{Vol}(D) = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx$$

$$= 8\int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dw\,dv\,du$$

(since D transforms to the solid unit ball $\{u^2+v^2+w^2 \le 1\}$)

$$= abc \cdot 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} dw\,dv\,du$$

$= abc\ \mathrm{Vol}($solid unit ball in (u,v,w)-coordinates$)$

$$= abc \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{4\pi}{3} abc$$

(spherical coordinates for (u,v,w) space)

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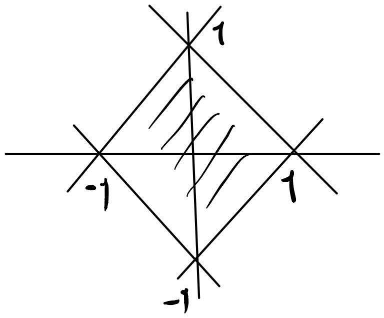
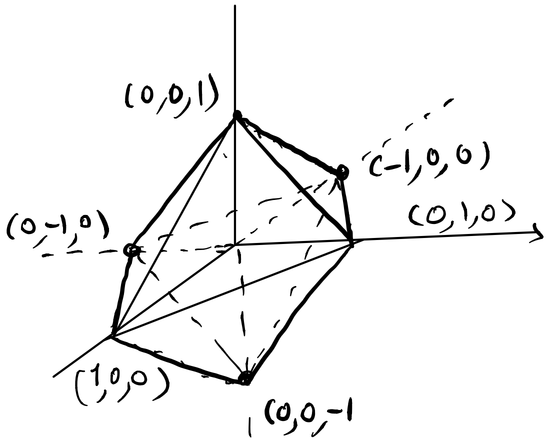
eg 31 Let $D = \{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \le 1\}$

Evaluate $\iiint_D (x+y+z)^4 \, dV$.

Soln: If $z=0$

then $|x|+|y| \le 1$





Boundary surfaces are given

$\pm x \pm y \pm z = 1$ (8 surfaces!)

Let $\begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases} \Leftrightarrow \begin{cases} x = \frac{1}{2}(u+w) \\ y = \frac{1}{2}(v-w) \\ z = \frac{1}{2}(u-v) \end{cases}$

Boundary planes:

$\pm x \pm y \pm z = 1$

$\begin{pmatrix} + & + & + \\ - & - & - \end{pmatrix}$ $\quad u = \pm 1$

$\begin{pmatrix} + & + & - \\ - & - & + \end{pmatrix}$ $\Longleftrightarrow$ $v = \pm 1$

$\begin{pmatrix} + & - & - \\ - & + & + \end{pmatrix}$ $\quad w = \pm 1$

$\begin{pmatrix} + & - & + \\ - & + & - \end{pmatrix}$ $\quad u - v + w = \pm 1$

(check!)

Change of variable famula $\Rightarrow$

$$\iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1\le u,v,w\le 1 \\ -1\le u-v+w\le 1}} u^4 \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| dv\,dw\,du$$

By $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \left|\det\begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix}\right| = \left|-\frac{1}{4}\right| = \frac{1}{4}$

$$\therefore \iiint_D (x+y+z)^4 \, dV = \iiint_{\substack{-1\le u,v,w\le 1 \\ -1\le u-v+w\le 1}} \frac{u^4}{4} \, dv\,dw\,du$$

$$= A - B - C$$

where $A = \iiint_{-1\le u,v,w\le 1} \frac{u^4}{4} \, dv\,dw\,du$

$$B = \iiint_{\substack{-1\le u,v,w\le 1 \\ u-v+w\ge 1}} \frac{u^4}{4} \, dv\,dw\,du$$

$$C = \iiint_{\substack{-1\le u,v,w\le 1 \\ u-v+w\le -1}} \frac{u^4}{4} \, dv\,dw\,du$$

$$A = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{u^4}{4}\,dv\,dw\,du \underset{}{\overset{\text{easy}}{=\!=}} \frac{2}{5} \quad (\text{check!})$$

	$u-v+w$
$(1,-1,1)$	3
$(1,1,1)$	1
$(-1,-1,1)$	1
$(1,-1,-1)$	1