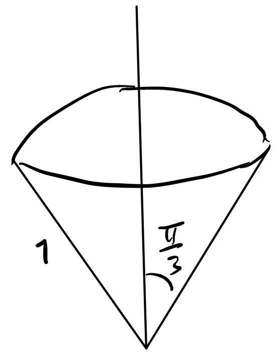
eg24 (see eg22)

Volume of ice-cream cone I again, in spherical coordinates



Soln : The ice-cream cone I is given by

$$\begin{cases} 0 \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{3} \\ 0 \le \theta \le 2\pi \end{cases}$$

don't miss this !

$$\Rightarrow \quad \mathrm{Vol}(I) = \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}} \sin\phi \, d\phi\right)\left(\int_0^1 \rho^2 d\rho\right)$$

$$= \frac{\pi}{3} \quad \text{(check!)}$$

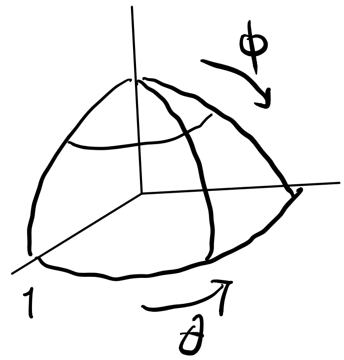
eg25 :

$$f(x,y,z) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}, & \text{if } (x,y,z) \ne (0,0,0) \\ 0, & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

(In fact, f is continuous, but it is sufficient to known f is continuous except the origin (0,0,0).)

Let D = unit ball centered at origin intersecting with the 1st octant

Then D can be represented in spherical coordinates:



$$\begin{cases} 0 \le \rho \le 1 \\ 0 \le \phi \le \frac{\pi}{2} \\ 0 \le \theta \le \frac{\pi}{2} \end{cases}$$

And for $(x,y,z) \neq (0,0,0)$,

$$f(x,y,z) = \frac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \frac{(\rho \sin\phi)^2}{\rho} = \rho \sin^2\phi$$

(as $\rho \to 0$, $f \to 0 \Rightarrow f$ is continuous at $(0,0,0)$)

Hence $$\iiint_D f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \underbrace{\rho \sin^2\phi}_{f} \cdot \rho^2 \underbrace{\sin\phi \, d\rho\, d\phi\, d\theta}_{\text{volume element}}$$

$$= \frac{\pi}{2} \left(\int_0^{\frac{\pi}{2}} \sin^3\phi\, d\phi\right)\left(\int_0^1 \rho^3 d\rho\right)$$

$$= \frac{\pi}{12} \quad \text{(check!)}$$

If we want to calculate the <u>average</u> of f over D, we need to calculate $Vol(D)$ too.

In our case $Vol(D) = \frac{1}{8} Vol(\text{unit sphere}) = \frac{1}{8} \cdot \frac{4\pi}{3} = \frac{\pi}{6}$

Hence <u>average of f over D</u> $= \frac{1}{Vol(D)} \iiint_D f(x,y,z)\,dV = \frac{1}{2}$

#

eg 26 : (Improper integrals)

Let $f(x,y,z) = \frac{1}{x^2+y^2+z^2} = \frac{1}{\rho^2}$ (unbounded as $\rho \to 0$)

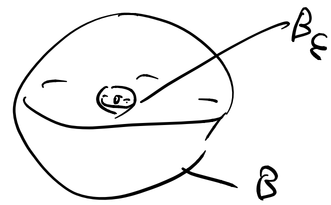
$$g(x,y,z) = \frac{1}{(\sqrt{x^2+y^2+z^2})^3} = \frac{1}{\rho^3}$$

over unit ball $B = \{(\rho,\phi,\theta) : 0 \le \rho \le 1\}$

(i) Does $\lim\limits_{\varepsilon \to 0} \iiint\limits_{B \setminus B_\varepsilon} f(x,y,z)\,dV$ exist?

where $B_\varepsilon = \{(\rho,\phi,\theta) : 0 \le \rho \le \varepsilon\}$

(ii) Does $\lim\limits_{\varepsilon \to 0} \iiint\limits_{B \setminus B_\varepsilon} g(x,y,z)\,dV$ exist?



Answer : For $f(x,y,z) = \frac{1}{x^2+y^2+z^2} = \frac{1}{\rho^2}$

$$\lim_{\varepsilon \to 0} \iiint\limits_{B \setminus B_\varepsilon} f(x,y,z)\,dV = \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \frac{1}{\rho^2} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon \to 0} \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi\, d\phi\right)\left(\int_\varepsilon^1 d\rho\right)$$

$$= \lim_{\varepsilon \to 0} 4\pi(1-\varepsilon) = 4\pi \quad \text{exists!}$$

For $g(x,y,z) = \frac{1}{(\sqrt{x^2+y^2+z^2})^3} = \frac{1}{\rho^3}$

$$\lim_{\varepsilon \to 0} \iiint\limits_{B \setminus B_\varepsilon} g(x,y,z)\,dV = \lim_{\varepsilon \to 0} \int_0^{2\pi} \int_0^{\pi} \int_\varepsilon^1 \frac{1}{\rho^3} \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \lim_{\varepsilon \to 0} \left(\int_0^{2\pi} d\theta \right) \left(\int_0^{\pi} \sin\phi \, d\phi \right) \left(\int_\varepsilon^1 \frac{1}{\rho} d\rho \right)$$

$$= \lim_{\varepsilon \to 0} 4\pi \ln \frac{1}{\varepsilon} \quad \text{doesn't exist !}$$

Terminology : we said that $f = \frac{1}{\rho^2}$ is "integrable" and

$g = \frac{1}{\rho^3}$ is "not integrable"

in the sense of improper integral.

Question : determine all $\beta > 0$ such that

$f = \frac{1}{\rho^\beta}$ is "integrable" in $B \subset \mathbb{R}^3$

Similar question in $\mathbb{R}^2$: determine all $\beta > 0$ such that

$f = \frac{1}{r^\beta}$ is "integrable" in $D \subset \mathbb{R}^2$, $D = \{r \leq 1\}$

(even in $\mathbb{R}^1$: $f = \frac{1}{|x|^\beta}$)

Application of Multiple integrals (Thomas' Calculus §15.6)

In applications, we often use the following:

In 2-dim: R is a region in $\mathbb{R}^2$ with density $\delta(x,y)$

- First moment about y-axis: $M_y = \iint_R x\,\delta(x,y)\,dA$
- First moment about x-axis: $M_x = \iint_R y\,\delta(x,y)\,dA$
- Mass: $M = \iint_R \delta(x,y)\,dA$
- Center of Mass (Centroid) $(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$

In 3-dim, D solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

- First moment:
  - about yz-plane: $M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$
  - about xz-plane: $M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$
  - about xy-plane: $M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$
- Mass: $M = \iiint_D \delta(x,y,z)\,dV$
- Center of Mass (Centroid) $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$

In 2-dim, R region in $\mathbb{R}^2$ with density $\delta(x,y)$

Moment of inertia

- about x-axis : $I_x = \iint_R y^2 \delta(x,y)\,dA$
- about y-axis : $I_y = \iint_R x^2 \delta(x,y)\,dA$
- about line L : $I_L = \iint_R r(x,y)^2 \delta(x,y)\,dA$

  where $r(x,y)$ = distance between (x,y) and L.

- about the origin $I_0 = \iint_R (x^2+y^2)\,\delta(x,y)\,dA$.

In 3-dim, D = solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

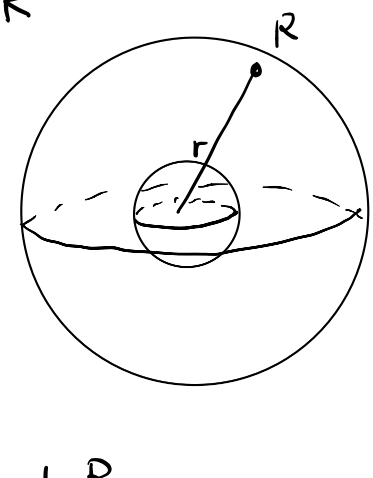
Moments of Inertia

- around x-axis : $I_x = \iiint_D (y^2+z^2)\,\delta(x,y,z)\,dV$
- around y-axis : $I = \iiint_D (x^2+z^2)\,\delta(x,y,z)\,dV$
- around z-axis : $I = \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$
- around line L : $I_L = \iiint_D r(x,y,z)^2\,\delta(x,y,z)\,dV$

  where $r(x,y,z)$ = distance between (x,y,z) and L.

eg 27 : Consider $D: \; r^2 \le x^2+y^2+z^2 \le R^2$

(r is a number , $0 < r < R$)



with density $\delta(x,y,z) \equiv \delta$

constant density function (i.e. uniform mass)

Express I_z in terms of

$m =$ mass of D, r, and R.

Soln : $I_z \overset{\text{def}}{=} \iiint\limits_D (x^2+y^2)\, \delta(x,y,z)\, dV$

$$= \delta \int_0^{2\pi}\int_0^{\pi}\int_r^R (\rho \sin\phi)^2 \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \delta \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin^3\phi\, d\phi\right)\left(\int_r^R \rho^4 d\rho\right)$$

$$= \frac{8\pi}{15}(R^5 - r^5)\cdot \delta$$

mass $m = \iiint\limits_D \delta(x,y,z)\, dV = \delta \iiint\limits_D dV$

$$= \delta \cdot \frac{4\pi}{3}(R^3 - r^3) \qquad \text{(check!)}$$

$$\Rightarrow \quad \boxed{I_z = \frac{2m}{5} \cdot \frac{R^5 - r^5}{R^3 - r^3}}.$$

Observation : Two limiting cases :

(i) $r \to 0$, i.e. the whole solid ball $\boxed{I_z = \frac{2m}{5} R^2}$

(ii) $r \to R$, i.e. a (hollow) sphere made of infinitesimally thin sheet.

$$\Rightarrow \quad I_z = \lim_{r \to R} \frac{2m}{5} \frac{R^5 - r^5}{R^3 - r^3} = \frac{2m}{5} \cdot \frac{5R^4}{3R^2} \quad \text{(check!)}$$

$$\boxed{I_z = \frac{2m}{3} R^2}$$

Moment of inertia of hollow sphere

> moment of inertia of the solid ball.

(assuming the same uniform mass) #

Change of Variables Formula

(Substitution in Multiple integrals)

Review of 1-variable

$$\int_a^b f(x)dx = \int_c^d \left[f(x(u))\frac{dx}{du}\right] du$$

$x = x(u)$ for $u \in [c, d]$

provided $\frac{dx}{du} > 0$ $(\Rightarrow c < d)$

and $\int_a^b f(x)dx = \int_d^c f(x(u))\frac{dx}{du}\, du$, if $\frac{dx}{du} < 0$

$(\Rightarrow c > d)$

Recall in Riemann sum (of general dimensions):

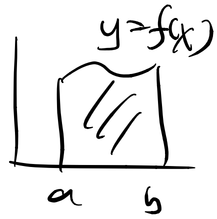
$$\int_{[a,b]} f(x)\, dx$$

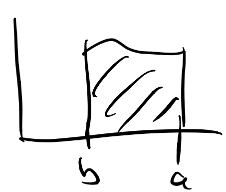
$dx \sim |\Delta x|$ length of Δx ! no direction !

as set (we don't care about the direction!)

$\therefore$ we actually have

$$\int_a^b f(x)dx = \begin{cases} \int_{[a,b]} f(x)dx & \text{if } a \le b \\ -\int_{[a,b]} f(x)dx, & \text{if } a \ge b \end{cases}$$





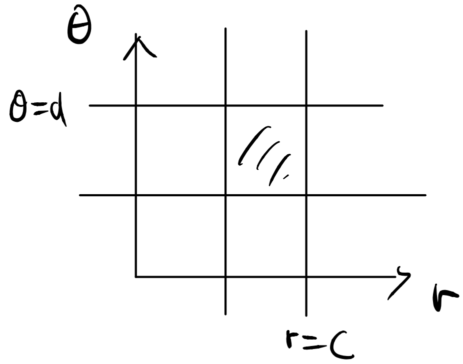
Combining these

$$\boxed{\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(x)\left|\frac{dx}{du}\right| du} \qquad \left(\text{since } \frac{|\Delta x|}{|\Delta u|} \sim \left|\frac{dx}{du}\right|\right)$$

$$\left(\frac{dx}{du}<0 \Rightarrow c>d \Rightarrow \int_{[c,d]} f(x)\left|\frac{dx}{du}\right|du = -\int_d^c f(x)\left|\frac{dx}{du}\right|du = \int_d^c f(x)\frac{dx}{du}\,du\right)$$

Back to multiple integrals

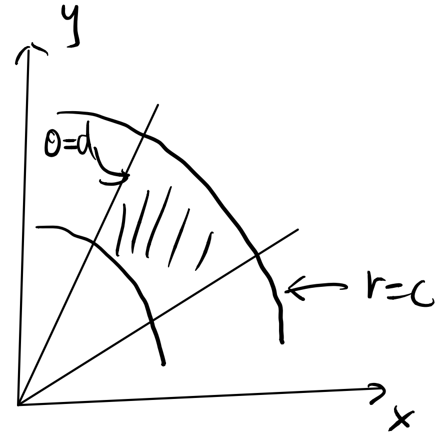
Recall: Polar coordinates

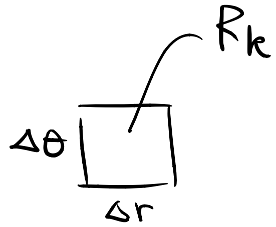


$$\xrightarrow{\phi}$$

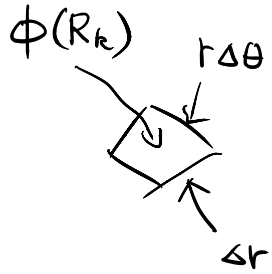
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\phi(r,\theta) = (x,y)$$





$$\xrightarrow{\phi}$$



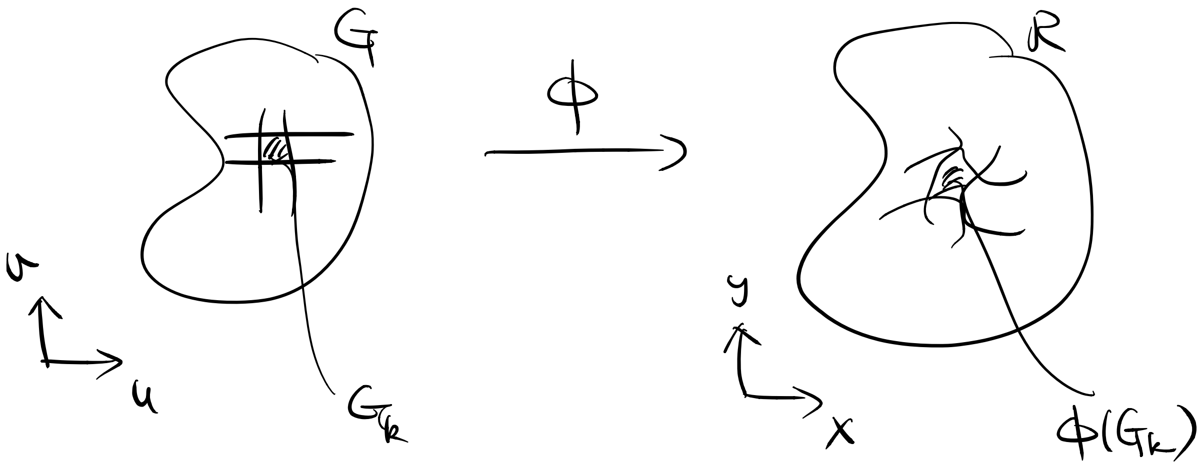
$$\text{Area}(R_k) \cong \Delta r\Delta\theta \qquad \text{Area}(\phi(R_k)) \cong r\Delta r\Delta\theta$$

$$\frac{\text{Area}(\phi(R_k))}{\text{Area}(R_k)} \to r \quad \text{as "}R_k \to \text{point"}.$$

General change of coordinate formula in $\mathbb{R}^2$

Suppose $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ denoted by

$\phi(u,v) = (x,y)$: $\phi : G \ (\subset uv\text{-plane}) \longrightarrow R \ (\subset xy\text{-plane})$



Idea: We need to find

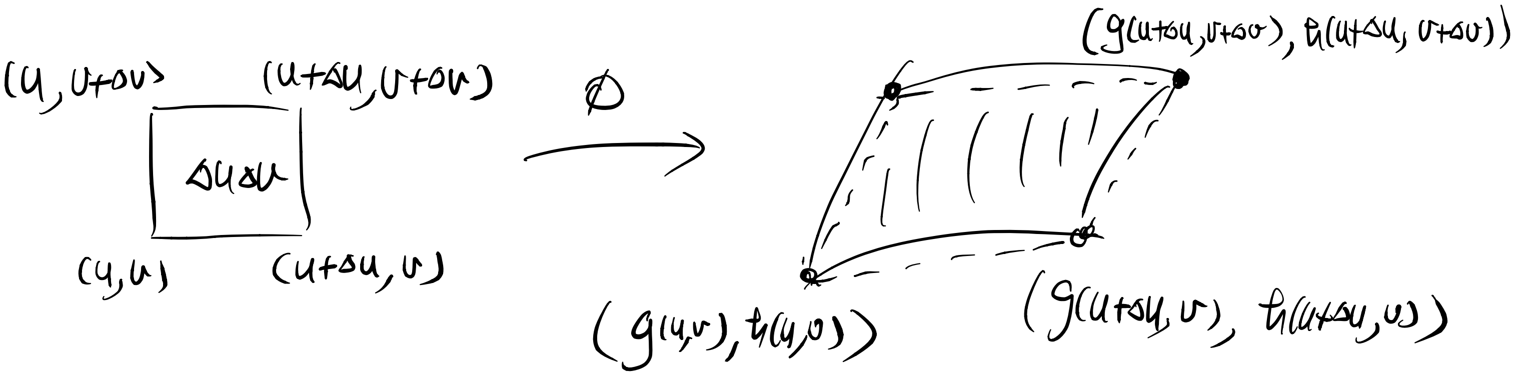
$$\frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \to \ ? \quad \text{as } \text{"}G_k \to \text{point"}$$

If ϕ is (diffeomorphism: 1-1, onto & $\phi, \phi^{-1} \in C^1$),

ϕ is $C^1 \Rightarrow$

$$\begin{cases} g(u+\Delta u, v+\Delta v) = g(u,v) + \frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v + \cdots \\ h(u+\Delta u, v+\Delta v) = h(u,v) + \frac{\partial g}{\partial u}\Delta u + \frac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

$$\Rightarrow \begin{cases} \Delta g = g(u+\Delta u, v+\Delta v) - g(u,v) = \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\[2ex] \Delta h = h(u+\Delta u, v+\Delta v) - h(u,v) = \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$



The "parallelogram" is "approximately" given by the linear transformation

$$\begin{pmatrix} \Delta g \\ \Delta h \end{pmatrix} = \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} \begin{pmatrix} \Delta u \\ \Delta v \end{pmatrix} + \cdots$$

(By linear algebra)

$$\Rightarrow \frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \cong \frac{\text{"}\Delta g \Delta h\text{"}}{\Delta u \Delta v} \cong \left| \det \begin{pmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{pmatrix} \right|.$$

"$\Delta g \Delta h$" = area of the parallelogram :