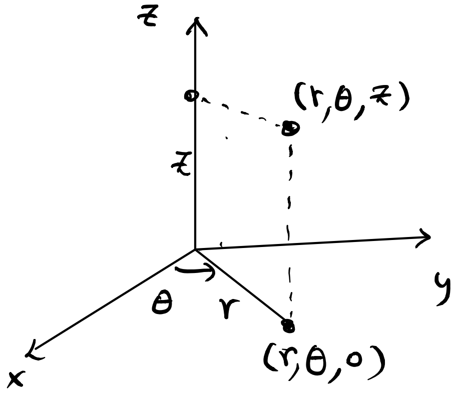
Cylindrical Coordinates in $\mathbb{R}^3$



- (r,θ) = polar coordinates for the xy-plane

  $(r \geq 0)$

- z = rectangular vertical coordinate

Then a point $P: (x,y,z)$ can be represented by (r,θ,z)

where
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

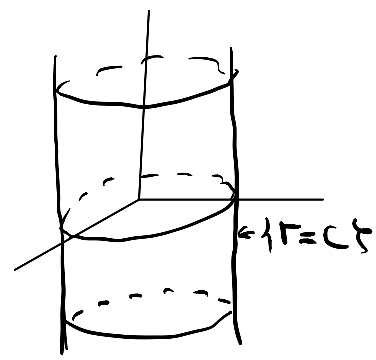
And (r,θ,z) is called the <u>cylindrical coordinates</u> for $\mathbb{R}^3$

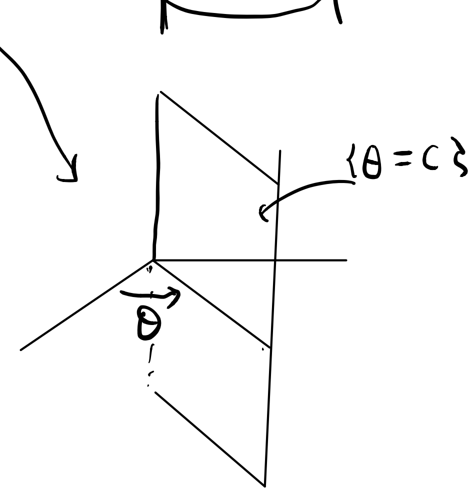
<u>Remark 1</u> : (Let c be a constant)

- $r = c$ $(c>0)$ describes a <u>cylinder</u>

- $\theta = c$ $(0 \leq c \leq 2\pi)$ describes a vertical half plane

- $z = c$ describes a horizontal plane (as in rectangular coordinates)





<u>Remark 2</u> : We can define cylindrical coordinates in other directions : <u>eg.</u>
$$\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$$

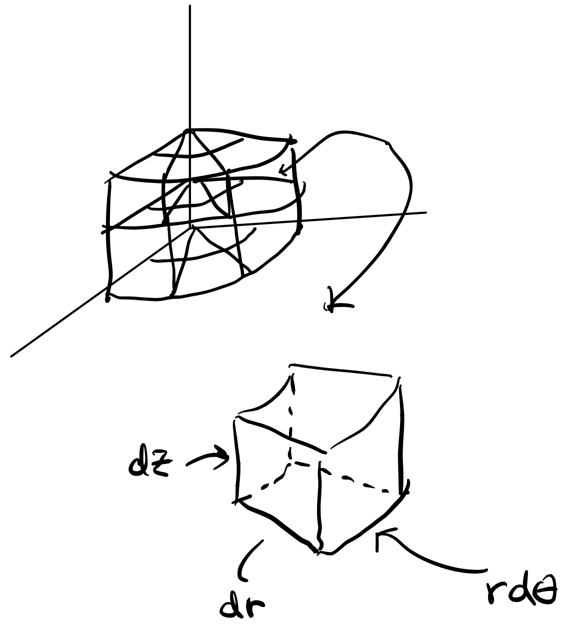
Volume element

$dV = \underbrace{dx\,dy}\,dz$

$\quad\;\downarrow \qquad \downarrow$

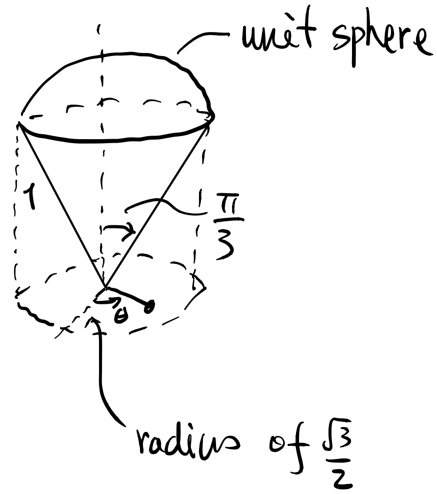
$= r\,dr\,d\theta\;dz$

(order of the integration can be changed)

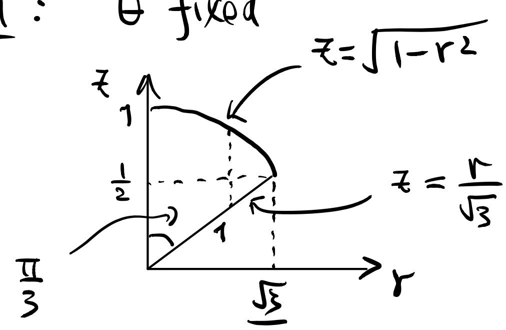


eg 22 (see also eg 24)

Find the volume of the Ice-cream cone I given as in figure.



Soln: θ fixed



Fubini's $\Rightarrow$ $\displaystyle Vol(D) = \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}}\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} \underline{r}\,dz\,dr\,d\theta$ don't miss this

$$= \int_0^{2\pi}\int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{\sqrt{3}}\right) r\,dr\,d\theta$$

$$= \frac{\pi}{3} \text{ (check!)}$$

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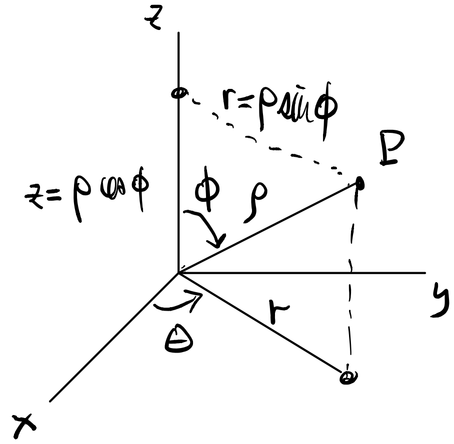
Spherical coordinates in $\mathbb{R}^3$



(ρ, ϕ, θ) where

- ρ = distance from the origin
  $(\rho \geq 0)$
- ϕ = angle from the positive z-axis to $\overline{OP}$
  $(0 \leq \phi \leq \pi)$
- θ = angle from cylindrical coordinates.
  $(0 \leq \theta \leq 2\pi)$

Remark: If (r, θ, z) is the cylindrical coordinates of point P

then $\begin{cases} r = \rho \sin\phi \\ z = \rho\cos\phi \end{cases}$

In particular $z^2 + r^2 = \rho^2$.

Then

$$\boxed{\begin{aligned} x &= r\cos\theta &&= \rho\sin\phi\cos\theta \\ y &= r\sin\theta &&= \rho\sin\phi\sin\theta \\ z &= z &&= \rho\cos\phi \end{aligned}}$$

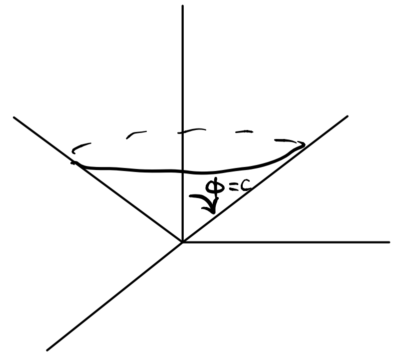
rectangular cylindrical spherical

Remark: If c is a constant, then

- $\rho = c$ $(c > 0)$ describes a sphere of radius c
- $\theta = c$ describes a vertical half-plane

• $\phi = c$ describes

$$= \begin{cases} +\text{ve } z\text{-axis}, & \text{if } c=0 \\ -\text{ve } z\text{-axis}, & \text{if } c=\pi \\ xy\text{-plane}, & \text{if } c=\frac{\pi}{2} \\ \text{cone} & \text{otherwise} \end{cases}$$

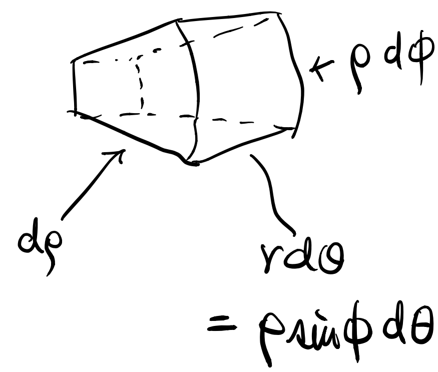


Volume element

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$= (\rho \sin\phi)\,\rho\,d\rho\,d\phi\,d\theta$$

i.e. $\boxed{dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}$



eg23 Convert the following into spherical coordinates

(1) $x^2+y^2+(z-1)^2=1$ (sphere)

(2) $z = -\sqrt{x^2+y^2}$ (cone)

Soln: (1) sub. $\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases}$

into $x^2+y^2+(z-1)^2=1$

$$\Leftrightarrow \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\Leftrightarrow \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi + 1 = 1$$

$$\Leftrightarrow \qquad \rho^2 = 2\rho\cos\phi$$

$$\Rightarrow \qquad \rho = 2\cos\phi \qquad \left(\begin{array}{l}\text{since } \rho \geq 0 \ \& \\ \rho = 0 \text{ is a point.}\end{array}\right)$$

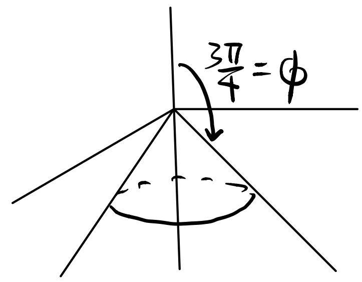
(2) Sub. the formula into $z = -\sqrt{x^2+y^2}$ $(= -r)$

$$\Rightarrow \qquad \rho\cos\phi = -\rho\sin\phi \qquad \left(\begin{array}{l}\rho \geq 0, \\ 0 \leq \phi \leq \pi \Rightarrow \sin\phi \geq 0\end{array}\right)$$

For $\rho \neq 0$ (i.e. not the origin)

$$\cos\phi = -\sin\phi$$

$$\Rightarrow \qquad \phi = \frac{3\pi}{4}$$



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