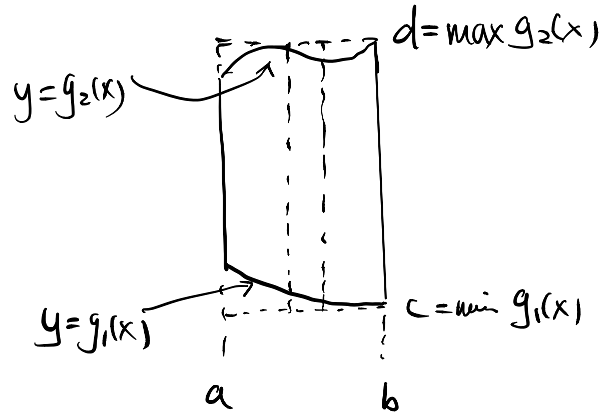
Pf: Type (1): Extend $f(x,y)$ to $F(x,y)$

as above on a rectangle

$R' = [a,b] \times [c,d]$ such that

$$c = \min_{[a,b]} g_1(x)$$

$$d = \max_{[a,b]} g_2(x)$$



By definition 2,

$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

$$\overset{\text{Fubini (1st form)}}{=} \int_a^b \left[\int_c^d F(x,y)\,dy \right] dx$$

[f continuous on $R \Rightarrow F$ continuous on R' except possibly on the boundary curve(s) of R. Hence by Prop 2', F (in fact $|F|$) is integrable over R'. And the Fubini theorem (1st form) is in fact true for "absolutely" integrable functions on a rectangle.]

Now $F(x,y) = 0$ for $y < g_1(x)$ and $y > g_2(x)$

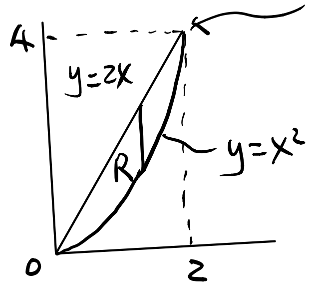
and $F(x,y) = f(x,y)$ for $g_1(x) \le y \le g_2(x)$.

$$\therefore \iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right] dx$$

Type (2) can be proved similarly. ※

eg 7 Integrable $f(x,y) = 4y+2$ over the region bounded by $y = x^2$ and $y = 2x$.

Soln



need to solve this point

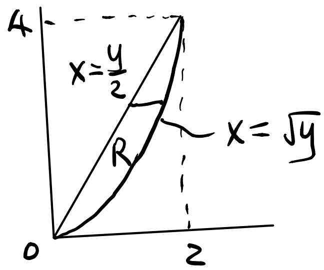
By Fubini's

$$\iint_R f(x,y)\,dA$$

$$= \int_0^2 \int_{x^2}^{2x} (4y+2)\,dy\,dx$$

$$= \int_0^2 (-2x^4 + 6x^2 + 4x)\,dx \quad \text{(check!)}$$

$$= \frac{56}{6} \qquad \text{(check!)}$$



In fact, R is also type (2), and Fubini's

$$\Rightarrow \iint_R f(x,y)\,dA = \int_0^4 \left[\int_{\frac{y}{2}}^{\sqrt{y}} (4y+2)\,dx \right] dy$$

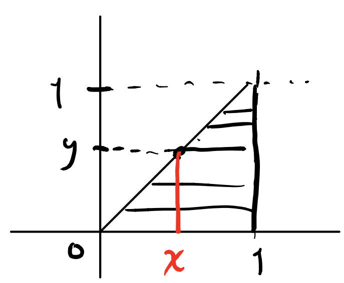
$$= \int_0^4 (4y+2)\left(\sqrt{y} - \frac{y}{2}\right) dy$$

$$= \cdots = \frac{56}{5} \quad \text{(check!)}$$

eg 8 : Evaluate $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$.

Soln : Regard $\int_0^1 \int_y^1 \frac{\sin x}{x}\,dx\,dy$ as a double integral of $\frac{\sin x}{x}$ over the region

$y \le x \le 1$ and $0 \le y \le 1$



By Fubini's

$$\int_0^1 \int_y^1 \frac{\sin x}{x}\, dx\, dy = \int_0^1 \int_0^x \frac{\sin x}{x}\, dy\, dx$$

$$= \int_0^1 \frac{\sin x}{x} \cdot x\, dx$$

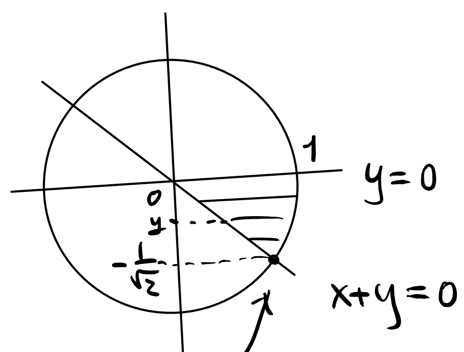
$$= \int_0^1 \sin x\, dx = 1 - \cos 1 \quad \#$$

(Caution: $f(x,y) = \frac{\sin x}{x}$ doesn't define at $x=0$,

why can we use Fubini? ($f(x,y) \geq 0$ & cts except on a line))

eg9 Find $\iint_R x\, dA$, where R is the region in the right half-plane bounded by $y=0$, $x+y=0$ and the unit circle.

Soln: Region R is as figure



By Fubini's

$$\iint_R x\, dA = \int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x\, dx\, dy$$

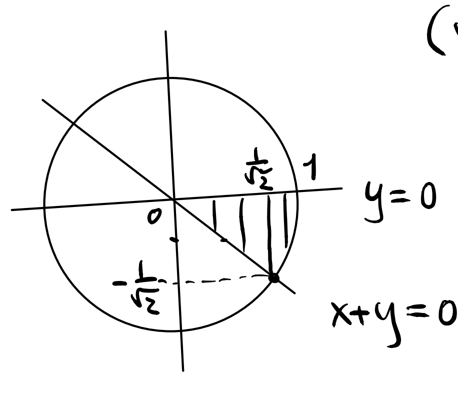
$$= \int_{-\frac{1}{\sqrt{2}}}^{0} \left(\tfrac{1}{2} - y^2\right) dy \quad \text{(check!)}$$

$$= \frac{1}{3\sqrt{2}} \quad \text{(check!)}$$

need to solve eqts. for this value

$$\begin{cases} x^2+y^2=1 \\ x+y=0 \end{cases} \Rightarrow y = -\frac{1}{\sqrt{2}}$$

(rejected $y = +\frac{1}{\sqrt{2}}$)



Alternatively

$$\iint_R x\,dA = \int_0^{\frac{1}{\sqrt{2}}}\int_{-x}^{0} x\,dy\,dx + \int_{\frac{1}{\sqrt{2}}}^{1}\int_{-\sqrt{1-x^2}}^{0} x\,dy\,dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x^2 dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2}\,dx = \frac{1}{3\sqrt{2}} \quad \text{(check!)}$$

Applications

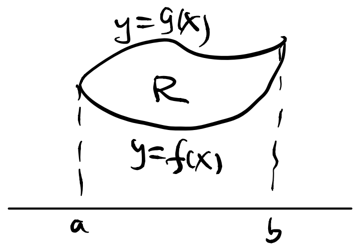
(1) Area (of (good) region $R \subset \mathbb{R}^2$)

$$\boxed{\textbf{Def 3} \quad \text{Area}(R) = \iint_R 1\,dA}$$

Then Fubini's Thm implies the well-known formula

$$\text{Area}(R) = \int_a^b [f(x) - g(x)]\,dx$$

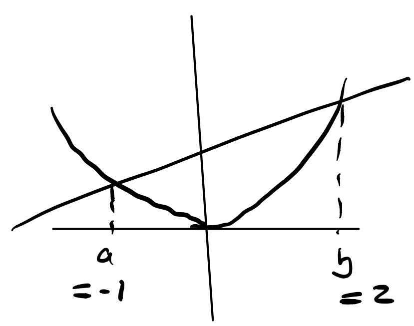
if R is the region bounded by the curves $y = f(x)$ and $y = g(x)$ ($f(a) = g(a)$, $f(b) = g(b)$, $f(x) \leq g(x)$) for $a \leq x \leq b$.



(Ex!)

eg 10: Area bounded by $y = x^2$ and $y = x+2$.

Soln : Solving $\begin{cases} y = x^2 \\ y = x+2 \end{cases} \Rightarrow x = -1, 2$



Then by Fubini's

$$\text{Area} = \int_{-1}^{2} (x+2-x^2)\,dx = \frac{9}{2} \quad (\text{check!})$$

(2) Average (of a function over a region)

Let $f: R \subset \mathbb{R}^2 \to \mathbb{R}$ be an integrable function

> **Def 4:** The average value of f over R
>
> $$= \frac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$$

eg 11: Let $f(x,y) = x\cos xy$, $R = [0,\pi]\times[0,1]$

Find average of f over R

Soln: Average of f over R $= \dfrac{1}{\text{Area}(R)} \iint_R f(x,y)\,dA$

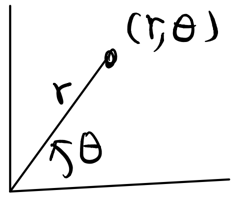
$$= \frac{1}{\pi}\int_0^{\pi}\int_0^{1} x\cos xy\,dy\,dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin x\,dx \quad (\text{check!})$$

$$= \frac{2}{\pi} \quad (\text{check!})$$

※

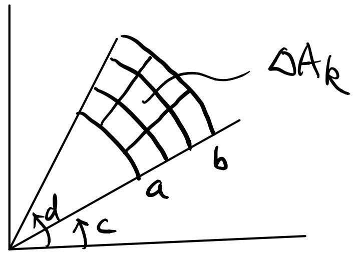
Double integral in polar coordinates



$(r,\theta) \leftrightarrow (x,y)$

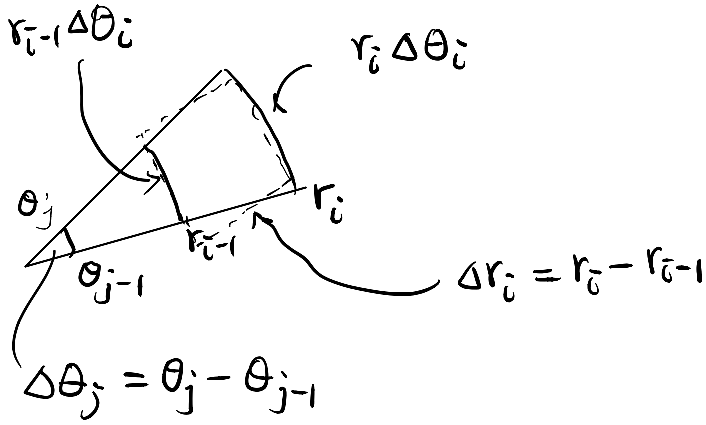
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} a \le r \le b \\ c \le \theta \le d \end{cases} \longrightarrow$$



Idea: $\sum_k f(\text{point}_k)\,\Delta A_k$

what is ΔA_k (approximately)?



$\therefore \quad \Delta A_k \simeq (r_i\Delta\theta_i)\,\Delta r_i \quad \left(\simeq (r_{i-1}\Delta\theta_i)\,\Delta r_i\right)$

Hence $\quad \Delta A_k \simeq \Delta x\,\Delta y \simeq \underline{(r\Delta\theta)}\,\Delta r$

So $\displaystyle\iint_R f(x,y)\,dA = \iint_R f(x,y)\,\underline{dx\,dy}$

$\displaystyle\qquad = \iint_R f(r\cos\theta, r\sin\theta)\,\underline{r\,dr\,d\theta}$

Method to remember the formula $\underline{dA = dx\,dy = r\,dr\,d\theta}$

Double integral of f over $R = \{(r,\theta) : a \le r \le b,\ c \le \theta \le d\}$ in polar coordinates is

$$\iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_c^d \left[\int_a^b f(r,\theta)\, r\,dr\right] d\theta$$

$$= \int_a^b \left[\int_c^d f(r,\theta)\, d\theta\right] r\,dr$$

where $f(r,\theta)$ is the simplified notation for $f(r\cos\theta, r\sin\theta)$.

Remark: This is a special case of the change of variables formula. The "extra" factor "r" in the integrand is in fact

$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

the Jacobian determinant of the change of coordinates.

More generally

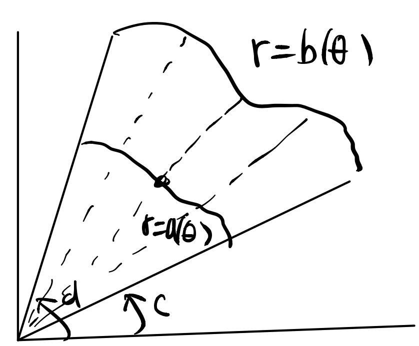
Thm 3: If R is a (closed and bounded) region with

$c \le \theta \le d$ and

$a(\theta) \le r \le b(\theta)$

$(0 \le a(\theta) \le b(\theta),\ a(\theta) \not\equiv b(\theta))$

And $f : R \to \mathbb{R}$ is a continuous function on R, then

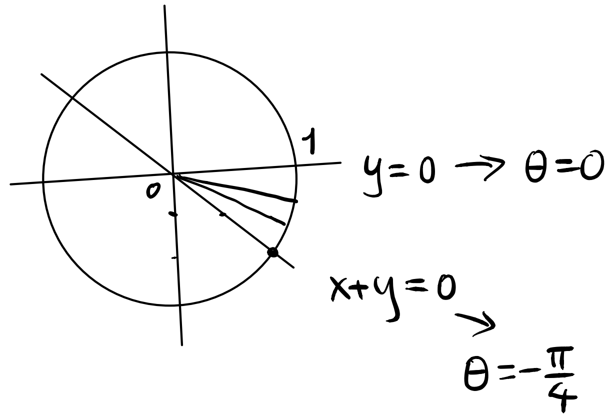


$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\right] d\theta$$

(remember this extra "r")

eg12: Back to our previous example 9.

$f(x,y) = x = r\cos\theta$



$$\int_{-\frac{1}{\sqrt{2}}}^{0} \int_{-y}^{\sqrt{1-y^2}} x\,dx\,dy$$

$$= \int_{-\frac{\pi}{4}}^{0} \left[\int_0^1 r\cos\theta \cdot r\,dr \right] d\theta$$

$$= \int_{-\frac{\pi}{4}}^{0} \left[\cos\theta \int_0^1 r^2\,dr \right] d\theta = \cdots = \frac{1}{3\sqrt{2}} \text{ (check!)}$$

Much easier than before!