Prop 4: (a) If $f(x,y) \geq 0$ is an integrable function on a closed rectangle R, then

$$\iint_R f(x,y)\,dA \geq 0$$

(b) If R_1 and R_2 be two closed rectangles such that $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \phi$, then

$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$

for integrable function over $R_1 \cup R_2$

Pf: Omitted (Obvious from the concept of Riemann sum)

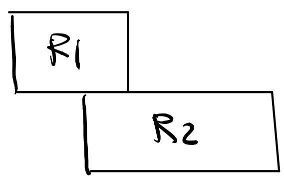
Note: Various situations for $\operatorname{int} R_1 \cap \operatorname{int} R_2 = \phi$

(1)

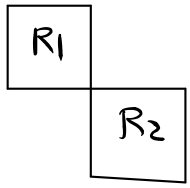
$R_1 \cap R_2$ = common edge

$\operatorname{int} R_1 \cap \operatorname{int} R_2 = \phi$

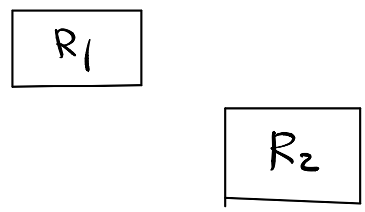
(2)



(3)



(4)

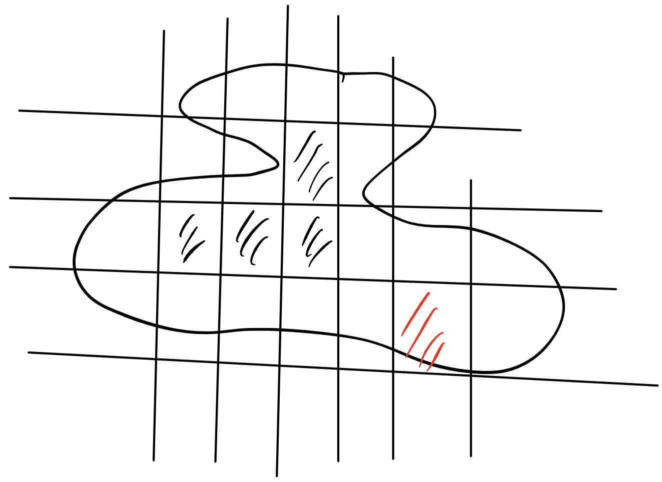


We haven't define $\iint_{R_1 \cup R_2} f(x,y)\,dA$ for cases (2) – (4).

Hence we need to define double integrals over general regions.

Double Integrals over General Regions

For non-rectangular bounded (closed) region R, one can define similarly the concept of "Riemann sum".

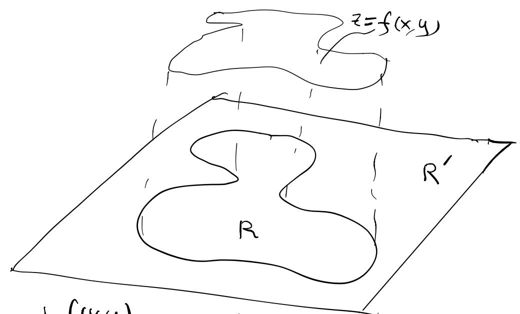


There are two ways to form the "sum":

(i) sum over all subrectangles completely inside R

(ii) sum over all subrectangles with non-empty intersection with R

Or define as follows:



$$F(x,y)=\begin{cases} f(x,y), & (x,y)\in R \\ 0, & (x,y)\in R'\setminus R \end{cases}$$

Def 2: let R be a bounded region and $f(x,y)$ be a function defined on R. For any rectangle $R'\supset R$, define

$$F(x,y)=\begin{cases} f(x,y), & \text{if } (x,y)\in R \\ 0, & \text{if } (x,y)\in R'\setminus R. \end{cases}$$

Then the integral of f over R is defined by

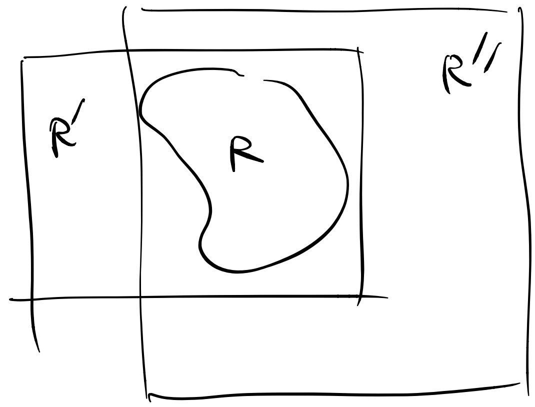
$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

Remark: The definition is well-defined (i.e. doesn't depend on the choice of R'): if R'' is another rectangle s.t. $R'' \supset R$

and $\tilde{F}(x,y) = \begin{cases} f(x,y), & \text{if } (x,y) \in R \\ 0, & \text{if } (x,y) \in R'' \setminus R \end{cases}$

Then (by Prop 4 (b))

$$\iint_{R''} \tilde{F}(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$



Prop 5: The propositions 1–4 hold if we replace "closed rectangle" by "closed and bounded region".

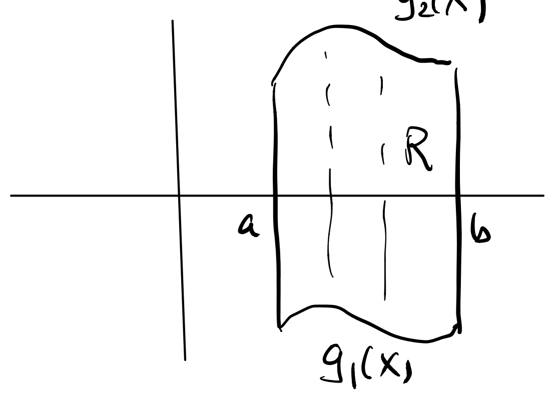
(together with the Prop 2')

Important special types of bounded regions R

Type (1) $R = \{(x,y) : a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$

where g_1 and g_2 are "continuous" functions on $[a,b]$

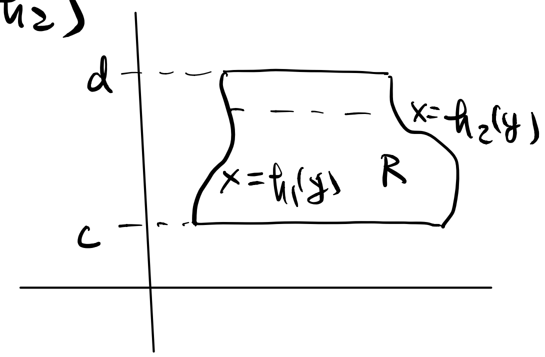
$(\ g_1 \leq g_2$ but $g_1 \not\equiv g_2)$



Type (2) $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

where h_1 and h_2 are "continuous" functions on $[c,d]$

($h_1 \le h_2$, but $h_1 \not\equiv h_2$)



For these 2 types of bounded regions, we have

Thm 2 (Fubini's Thm (Stronger version))

Let $f(x,y)$ be a continuous function on a closed and bounded region R.

(1) If R is of type (1) as above, then

$$\iint_R f(x,y)\,dA = \int_a^b \left[\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\right] dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(2) If R is of type (2) as above, then

$$\iint_R f(x,y)\,dA = \int_c^d \left[\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\right] dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$