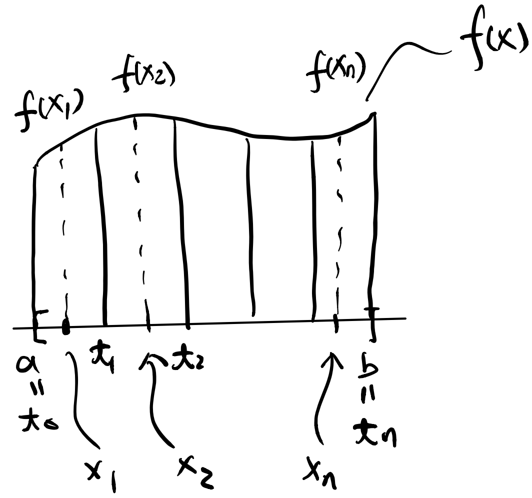
Double Integrals

Recall: In one-variable, "integral" is regarded as "limit" of "Riemann sum" (take MATH 2060 for rigorous treatment).

$$\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

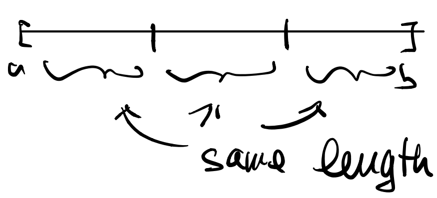
where
$$\begin{cases} f \text{ is a function on the interval } [a,b] \\ P \text{ is a } \underline{\text{partition}} \quad a = t_0 < t_1 < \cdots < t_n = b \\ x_k \in [t_{k-1}, t_k] \text{ and } \Delta x_k = t_k - t_{k-1} \\ \|P\| = \max_k |\Delta x_k| \end{cases}$$



Remark: We usually use uniform partition P

$$a = t_0 < t_1 = a + \frac{1}{n}(b-a) < t_2 = a + \frac{2}{n}(b-a) < \cdots$$

$$\cdots < t_k = a + \frac{k}{n}(b-a) < \cdots = t_n = b$$

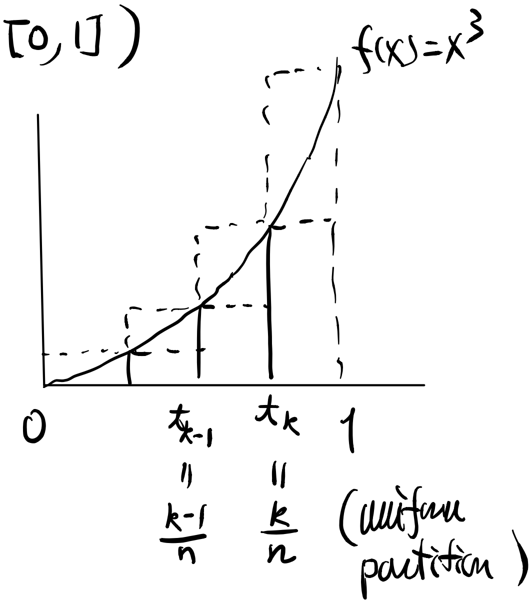


In this case, $\|P\| = \max_k |\Delta x_k| = \frac{b-a}{n} \to 0 \iff n \to \infty$.

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\,\frac{b-a}{n}$$

eg1: Find $\int_0^1 x^3\,dx$ (i.e. $f(x)=x^3$ on $[0,1]$)



Soln. (1) One may choose

$$x_k = \frac{k-1}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$$

then
$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$

$$= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 \cdot \frac{1}{n}$$

$$= \frac{1}{n^4} \sum_{k=1}^{n} (k-1)^3$$

$$= \frac{1}{n^4} \cdot \frac{(n-1)^2 n^2}{4} \quad \text{(Check!)}$$

$$= \frac{1}{4}\left(1-\frac{1}{n}\right)^2$$

$$\to \frac{1}{4} \quad \text{as } n\to\infty$$

$$\therefore \int_0^1 x^3\,dx = \frac{1}{4}.$$

(2) Or we can choose $x_k = \frac{k}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$

(Will we get different answer?)

Then $S_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \cdot \frac{1}{n}$ ($x_k = \frac{k}{n}$)

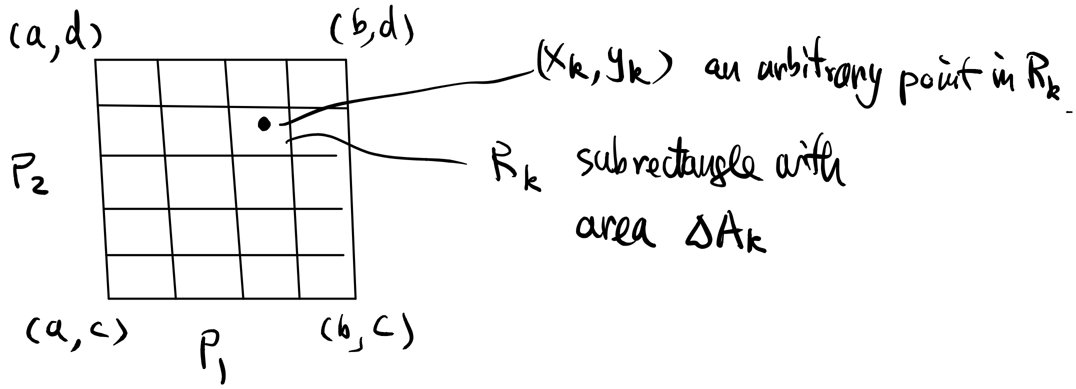
$$= \frac{1}{n^4} \frac{n^2(n+1)^2}{4} = \frac{1}{4}\left(1+\frac{1}{n}\right)^2 \to \frac{1}{4} \quad \text{as } n \to \infty$$

(same limit)

<u>Remark</u>: We can use any $x_k \in [t_{k-1}, t_k]$ and still get the same $\int_0^1 x^3 dx = \frac{1}{4}$.

This concept can be generalized to <u>any dimension</u>.

For 2-dim., let we first consider a function $f(x,y)$ defined on a rectangle $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$



Then we can subdivide R into sub-rectangles by using partitions P_1 of $[a,b]$ and P_2 of $[c,d]$.

Denote $P = P_1 \times P_2$ (partition, subdivision, of R)

and $\|P\| = \max(\|P_1\|, \|P_2\|)$.

Let the sub-rectangles be R_k, $k = 1, \cdots, N$ = number of subrectanges

with <u>areas</u> ΔA_k.

Choose point $(x_k, y_k) \in R_k$ (for each k), then

consider the "Riemann sum"

$$S(f,P)=\sum_{k=1}^{N} f(x_k,y_k)\,\Delta A_k$$

Def 1: The function f is said to be integrable over R

if $$\lim_{\|P\|\to 0} S(f,P)=\lim_{\|P\|\to 0}\sum_{k=1}^{N} f(x_k,y_k)\,\Delta A_k$$

exists and independent of the choice of $(x_k,y_k)\in R_k$.

In this case, the limit is called the (double) integral of f over R and is denoted by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$