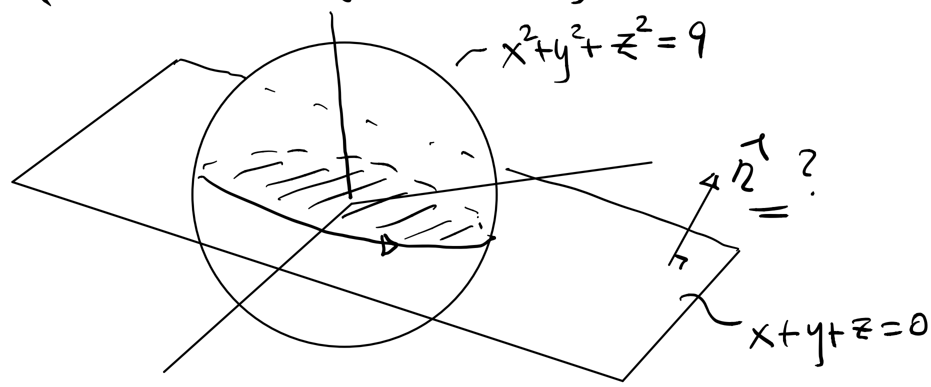
eg62 Let $\vec{F} = y\hat{i} - x\hat{j}$ (same $\vec{F}$ as in eg61, new surface & new boundary curve)



$$S_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \le 9,\ x+y+z=0\}$$

boundary curve of S_4 : $C_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 9,\ x+y+z=0\}$

Find $\oint_{C_4} \vec{F} \cdot d\vec{r}$ (with direction of C_4 given as in the figure.)

Soln: Apply Stokes' Thm

$$\oint_{C_4} \vec{F} \cdot d\vec{r} = \iint_{S_4} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

$\hat{n}$ is the normal to S_4 is given the coefficient of the eqt. of the plane

ie. $\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$ + → upward

$$= \iint_{S_4} (-2\hat{k}) \cdot \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\, d\sigma$$

$$= -\frac{2}{\sqrt{3}} \iint_{S_4} d\sigma = -\frac{2}{\sqrt{3}} \mathrm{Area}(S_4)$$

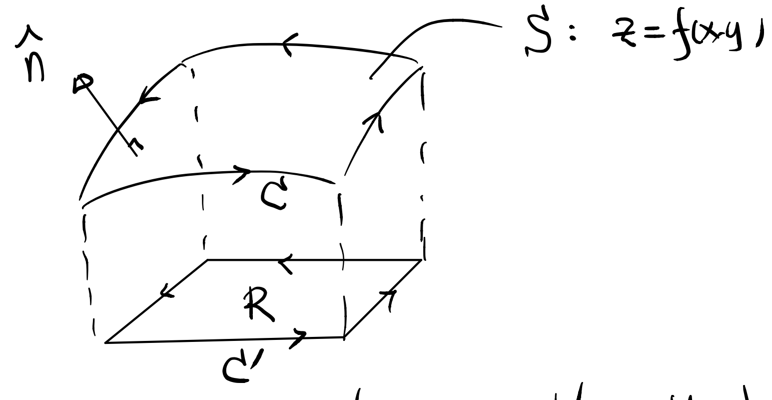
$$= -\frac{2}{\sqrt{3}} (\pi \cdot 3^2) = -\frac{18\pi}{\sqrt{3}}$$

#

Proof of Stokes' Thm

Special case: S is a graph given by

$z = f(x,y)$ over a region R with upward normal



Assume C is the boundary of S, and C' is the boundary of R (anti-clockwisely oriented wrt the normal of S and the plane respectively)

Parametrize the graph as

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}, \quad (x,y) \in R$$

Then as before
$$\begin{cases} \vec{r}_x = \hat{i} + \frac{\partial f}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \frac{\partial f}{\partial y}\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k} \quad \text{(upward)}$$

Hence $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|}$ is the upward normal of S,

and $d\sigma = |\vec{r}_x \times \vec{r}_y|\, dx\,dy = |\vec{r}_x \times \vec{r}_y|\, dA$

dA area element of R

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ be the C^1 vector field.

Then $$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = \iint_R (\vec{\nabla}\times\vec{F})(\vec{r}(x,y))\cdot\frac{\vec{r}_x\times\vec{r}_y}{\cancel{|\vec{r}_x\times\vec{r}_y|}}\cdot\cancel{|\vec{r}_x\times\vec{r}_y|}\,dA$$

$$= \iint_R \left[(L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k}\right]\cdot\left[-\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}\right]dA$$

$$= \iint_R \left[-f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y)\right]dx\,dy$$

For the line integral

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy + L\,dz$$

parametrized by $(x,y)\in C'$

$$= \oint_{C'} M\,dx + N\,dy + L\,df \qquad (\quad z = f(x,y) \quad)$$

$$= \oint_{C'} M\,dx + N\,dy + L(f_x\,dx + f_y\,dy)$$

$$= \oint_{C'} (M + Lf_x)\,dx + (N + Lf_y)\,dy.$$

Precisely: If C' is parametrized by
$\vec{\gamma}(t) = (x(t), y(t))$ for $a \le t \le b$.

Then C is parametrized by

$$\vec{r}(t) = (x(t), y(t), f(x(t), y(t)))$$

$$= x(t)\hat{i} + y(t)\hat{j} + f(x(t), y(t))\hat{k}, \quad a \le t \le b$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_a^b \left[\begin{aligned} &M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) \\ &+ L(\vec{r}(t))\,\frac{d}{dt} f(x(t),y(t)) \end{aligned} \right] dt$$

$$= \int_a^b \left[Mx' + Ny' + L(f_x x' + f_y y') \right] dt$$

$$= \int_a^b \left[(M + Lf_x)x' + (N + Lf_y)y' \right] dt$$

$$= \oint_{C'} (M + Lf_x)\,dx + (N + Lf_y)\,dy$$

Then by Green's Thm

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C'} (M + Lf_x)\,dx + (N + Lf_y)\,dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(N + Lf_y) - \frac{\partial}{\partial y}(M + Lf_x) \right] dA$$

$$= \iint_R \left\{ \begin{aligned} &\frac{\partial}{\partial x}\left[N(x,y,f(x,y)) + L(x,y,f(x,y))\,f_y(x,y) \right] \\ &- \frac{\partial}{\partial y}\left[M(x,y,f(x,y)) + L(x,y,f(x,y))\,f_x(x,y) \right] \end{aligned} \right\} dA$$

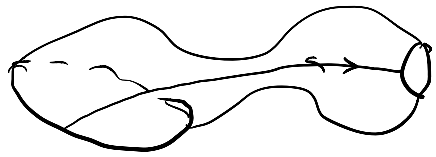
$$= \iint_R \left[\begin{aligned} &(N_x + N_z f_x) + (L_x + L_z f_x) f_y + L\cancel{f_{yx}} \\ &- \left[(M_y + M_z f_y) + (L_y + L_z f_y) f_x + L\cancel{f_{xy}} \right] \end{aligned} \right] dA$$

$$= \iint_R \left[-f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y) \right] dA$$

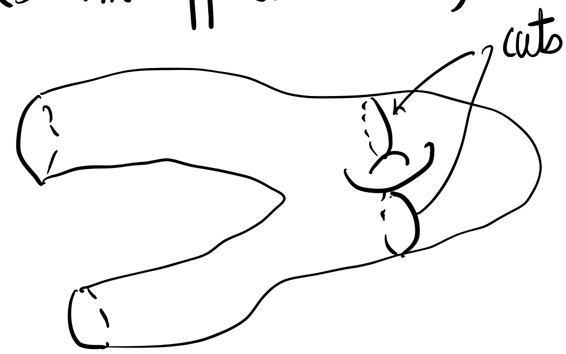
$$= \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$ This proves the case of C^2 graph.

General case: Divides S into finitely many pieces which are graphs (in certain projection).

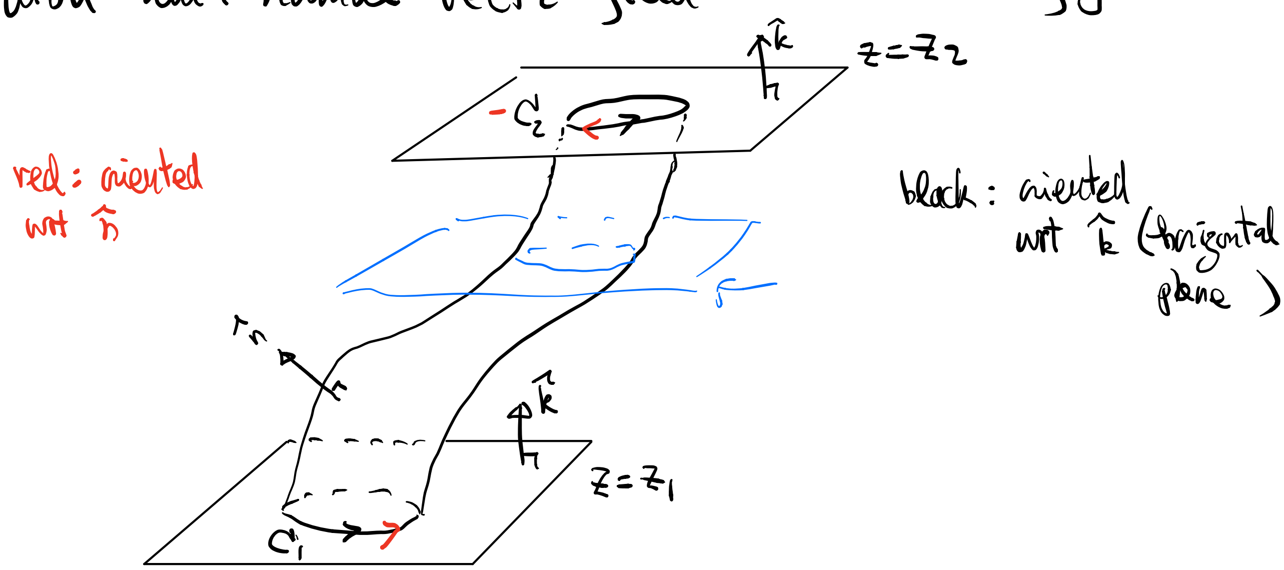
[This includes S with many boundary components as in the Green's Thm


]

Note: Stokes' Thm applies to surfaces like the following



eg 63: Let $\vec{F}$ be a vector field such that $\vec{\nabla}\times\vec{F}=0$ and defined on a region containing the surface S with unit normal vector field $\hat{n}$ as in the figure:



The boundary C of S has 2 components C_1 and C_2 at the level $z=z_1$ & $z=z_2$ respectively.

If both C_1, C_2 oriented anticlockwisely with respect to the "horizontal planes".

Then when C oriented with respect to $\hat{n}$ (the surface normal)

we have $\quad C = C_1 - C_2$

And Stokes' Thm $\Rightarrow$

$$0 = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = \oint_C \vec{F}\cdot d\vec{r}$$

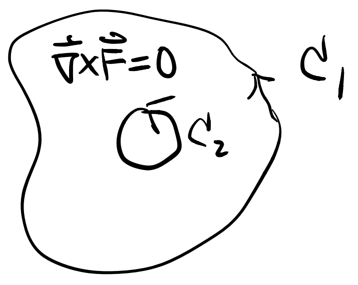
oriented wrt $\hat{n}$.

$$= \oint_{C_1} \vec{F}\cdot d\vec{r} - \oint_{C_2} \vec{F}\cdot d\vec{r}$$

oriented wrt "horizontal plane"

$$\Rightarrow \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot d\vec{r}$$

Compare this with Green's Thm on plane region with one hole:



$$\Rightarrow \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot d\vec{r} \quad \text{(check!)}$$

anti-clockwisely wrt "plane". (not see region as a surface)

Proof of Thm 10 (3-dim'l case)

Only the "$\Leftarrow$" part remains to be proved:

By assumption $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ satisfies the system of eqts in the cor. to the Thm 9, that is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial L}{\partial y}, \quad \text{and} \quad \frac{\partial L}{\partial x} = \frac{\partial M}{\partial z}.$$

Hence $\vec{\nabla} \times \vec{F} = 0$

Let C be a simple closed curve in a simply-connected region D

Then C can be deformed to a point inside D

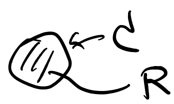
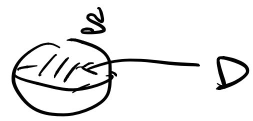
The process of deformation gives an oriented surface $S \subset D$ such that

the boundary of $S = C$.

By Stokes' Thm,
$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma$$
$$= 0 \quad (\text{since } \vec{\nabla} \times \vec{F} = 0)$$

Then Thm 9 $\Rightarrow$ $\vec{F}$ is conservative. ##

Summary

$n=2$	$n=3$
Tangential form of Green's Thm	**Stokes' Thm**
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA$$	$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$
Normal form of Green's Thm	**Divergence Thm** (next topic)
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{\nabla}\cdot\vec{F}\,dA$$	$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\,dV$$
	
"flux": by definition $\hat{n}$ is the "outward" normal of curve "C" in the "plane".	surface normal of S, "outward" pointing which can be defined when S encloses a solid region D

Thm 13 (Divergence Theorem)

Let $\vec{F}$ be a C^1 vector field on $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^3$

no boundary

$\underset{\sim}{S}$ be a piecewise smooth oriented closed surface

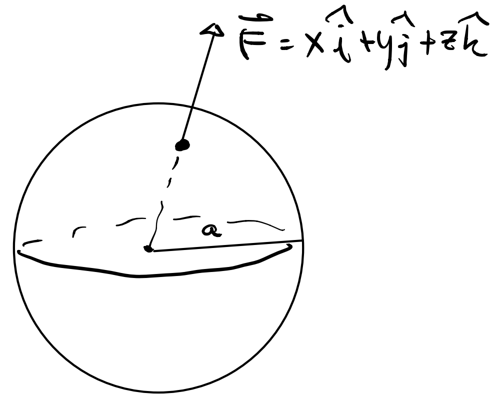
enclosing a (solid) region $D \subseteq \Omega$.

Let $\hat{n}$ be the outward pointing unit normal vector field on $\underset{\sim}{S}$.

Then

$$\boxed{\iint_{\underset{\sim}{S}} \vec{F} \cdot \hat{n}\, d\sigma = \iiint_D \operatorname{div} \vec{F}\, dV = \iiint_D \vec{\nabla} \cdot \vec{F}\, dV}$$

eg 64 Verify Divergence Thm for



$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\underset{\sim}{S} = \{x^2 + y^2 + z^2 = a^2\} \quad (a > 0)$$

(sphere $= S^2_a$ 2-dim'l sphere of radius a centered at $(0,0,0)$)

D = solid ball bounded by $\underset{\sim}{S}$.

Soln: At $(x,y,z) \in \underset{\sim}{S}$

$$\hat{n} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})$$ is the outward pointing unit normal

$$\iint_{\underset{\sim}{S}} \vec{F} \cdot \hat{n}\, d\sigma = \iint_{\underset{\sim}{S}} (x\hat{i} + y\hat{j} + z\hat{k}) \cdot \frac{1}{a}(x\hat{i} + y\hat{j} + z\hat{k})\, d\sigma$$

$$= \iint_S a\, d\sigma = a\,\text{Area}(S) = 4\pi a^3 \quad (\text{check!})$$

On the other hand

$$\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\Rightarrow \iiint_D \text{div}\vec{F}\,dV = \iiint_D 3\,dV = 3\,\text{Vol}(D) = 3\cdot\frac{4\pi a^3}{3} = 4\pi a^3$$

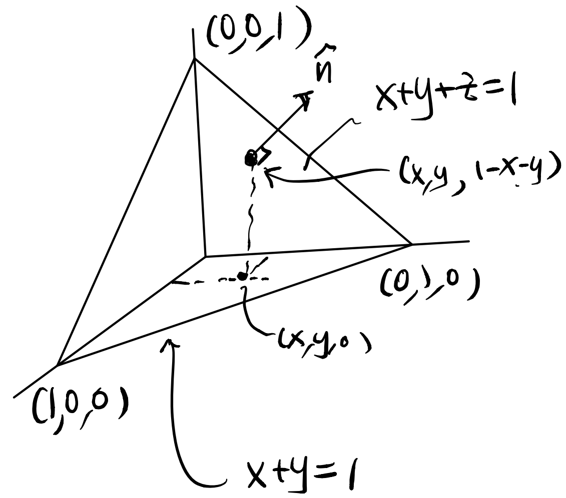
$$= \iint_S \vec{F}\cdot\hat{n}\,d\sigma \qquad \#$$

eg65 $\vec{F} = x\sin y\,\hat{i} + (\cos y + z)\hat{j} + z^2\hat{k}$

Compute outward flux of $\vec{F}$ across the boundary ∂T of

$$T = \left\{(x,y,z)\in\mathbb{R}^3 : \begin{array}{l} x+y+z\le 1 \\ x,y,z\ge 0 \end{array}\right\}$$

(tetrahedron)



Soln: $\text{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y + z) + \frac{\partial}{\partial z}(z^2)$

$$= 2z \quad (\text{check!})$$

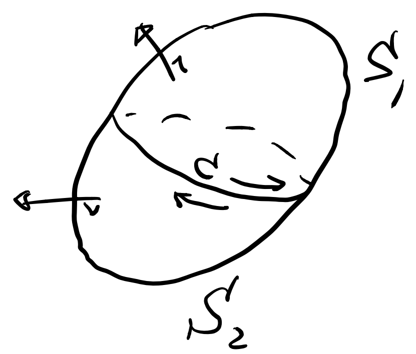
Divergence Thm

$$\Rightarrow \iint_{\partial T} \vec{F}\cdot\hat{n}\,d\sigma = \iiint_T \text{div}\vec{F}\,dV$$

$$= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 2z \, dz \, dy \, dx$$

$$= \frac{1}{12} \quad \text{(check!)}$$

eg66: Let S_1, S_2 be 2 surfaces with common boundary curve C such that $S_1 \cup S_2$ forms a closed surface enclosing a solid region D (without hole)



Suppose $\hat{n}$ is the outward normal of the solid region D. Then the orientation of C with respect to $(S_1, \hat{n})$ and $(S_2, \hat{n})$ are opposite

(since "$\hat{b}$" pointing to opposite side)

Stokes' Thm $\Rightarrow$

$$\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma = \oint_C \vec{F} \cdot d\vec{r} \quad \text{(+ve oriented wrt } (S_1, \hat{n}))$$

$$= - \oint_C \vec{F} \cdot d\vec{r} \quad \text{(+ve oriented wrt } (S_2, \hat{n}))$$

$$= - \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma$$

$$\Rightarrow \iint_{S_1 \cup S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma = 0$$

(see eg61(c) for explicit example)

Divergence Thm $\Rightarrow$

$$\iiint_D \operatorname{div}(\vec{\nabla}\times\vec{F})\,dV = \iint_{S_1 \cup S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = 0.$$

(true for "any" C^2 vector field $\vec{F}$ defined on "any" D)

It is consistent with $\boxed{\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0}$ (Ex!)

$\forall$ C^2 vector field

i.e. $\boxed{\operatorname{div}(\operatorname{curl}\vec{F}) = 0}$

Compare : $\boxed{\operatorname{curl}(\operatorname{grad} f) = 0 \Leftrightarrow \vec{\nabla}\times(\vec{\nabla} f) = 0}$