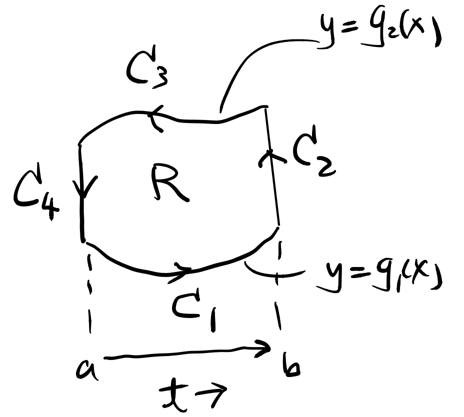
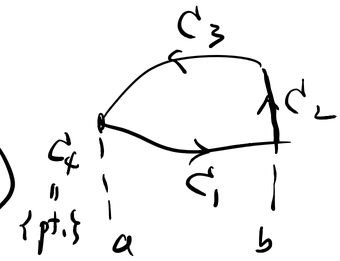
Pf of Green's Thm for Simple Region



By definition, R is of type (1) and can be written as

$$R=\{(x,y): a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$$

Let denote the components of the boundary of R by C_1, C_2, C_3, and C_4 as in the figure (Note: C_2 and/or C_4 could just be a point)



Then $\partial R = C_1+C_2+C_3+C_4$ as oriented curve

(using "+" instead of "∪" to denote the orientation)

Now $C_1=\{y=g_1(x)\}$ can be parametrized by

$$(x,y)=\vec{r}(t)=(t,g_1(t)),\quad a\le t\le b$$

with correct orientation.

$$\therefore\ \int_{C_1} M\,dx=\int_a^b M(t,g_1(t))\,dt$$

Similarly "$-C_3$" can be parametrized by

$$\vec{r}(t)=(t,g_2(t)),\quad a\le t\le b$$

with correct orientation

$$\therefore\ \int_{-C_3} M\,dx=\int_a^b M(t,g_2(t))\,dt$$

$$\Rightarrow\ \int_{C_3} M\,dx=-\int_{-C_3} M\,dx=-\int_a^b M(t,g_2(t))\,dt$$

For $C_2 = \{x=b\}$, it can be parametrized by

$$\vec{r}(t) = (b, t) \quad , \quad g_1(b) \le t \le g_2(b)$$

with correct orientation

$$\Rightarrow \quad \int_{C_2} M\,dx = 0 \qquad \left(\text{since } \frac{dx}{dt} \equiv 0\right)$$

Similarly $\int_{C_4} M\,dx = -\int_{-C_4} M\,dx = 0$

Hence

$$\oint_{\partial R} M\,dx = \sum_{i=1}^{4} \int_{C_i} M\,dx$$

$$= \int_a^b \left[M(t, g_1(t)) - M(t, g_2(t))\right] dt$$

$$\left(= \int_a^b \left[M(x, g_1(x)) - M(x, g_2(x))\right] dx\right)$$

On the other hand, Fubini's Thm $\Rightarrow$

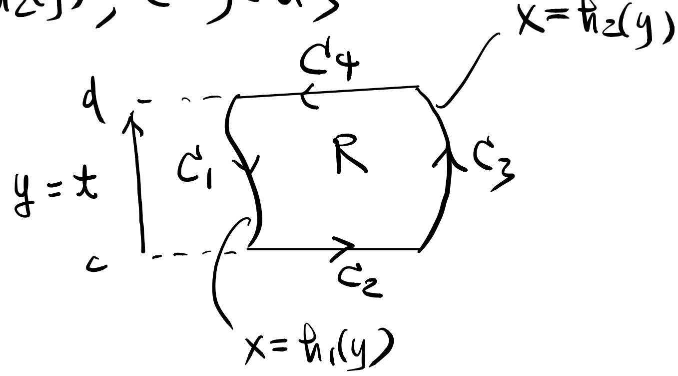
$$\iint_R -\frac{\partial M}{\partial y}\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} -\frac{\partial M}{\partial y}\,dy\right) dx$$

$$= \int_a^b -\left[M(x, g_2(x)) - M(x, g_1(x))\right] dx$$

$$= \oint_{\partial R} M\,dx$$

Similar, R is also typo (2), R can be written as

$$R = \{ (x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d \}$$



$$\oint_{\partial R} N\,dy = -\int_c^d N(h_1(t),t)\,dt$$

$$+\ 0$$

$$+ \int_c^d N(h_2(t),t)\,dt$$

$$+\ 0$$

$$= \int_c^d [N(h_2(t),t) - N(h_1(t),t)]\,dt$$

$$= \int_c^d [N(h_2(y),y) - N(h_1(y),y)]\,dy$$

$$= \int_c^d \left(\int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\,dx \right) dy$$

$$= \iint_R \frac{\partial N}{\partial x}\,dA$$

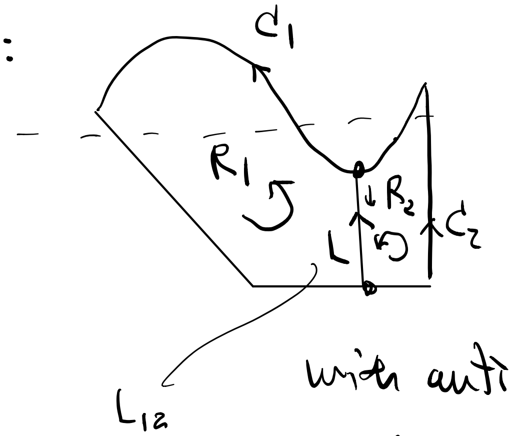
All together $\displaystyle \oint_{\partial R} (M\,dx + N\,dy) = \iint_R -\frac{\partial M}{\partial y}\,dA + \iint_R \frac{\partial N}{\partial x}\,dA$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \quad . \#$$

Proof of Green's Thm for

R = finite union of simple regions with intersections only along some boundary line segments, and those line segments touch only at the end points at most.

eg:

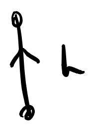


R_1, R_2 = simple

but $R = R_1 \cup R_2 \neq$ simple

$\partial R_1 = C_1 + L$

$\partial R_2 = C_2 - L$



with anti-clockwise orientation

and $\partial R = C_1 + C_2$

By assumption $R = \cup R_i$ finite union s.t.

- R_i are simple, and
- $R_i \cap R_j$ = line segment of a common boundary portion denoted by L_{ij} $(i \neq j)$ (may be empty)

Then $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \sum_i \oint_{\partial R_i} M dx + N dy \quad \left(\begin{array}{l}\text{by Green's Thm} \\ \text{for simple region}\end{array}\right)$$

Denote C_i = the part of ∂R_i with no intersection with any other R_j (except at the end points)

Then
$$\partial R_i = C_i + \sum_{\substack{j \\ (j\neq i)}} L_{ij}$$

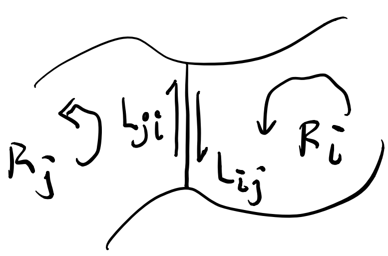
where L_{ij} is oriented according to the anti-clockwise orientation of ∂R_i

Hence
$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \oint_{C_i + \sum_{\substack{j \\ (j\neq i)}} L_{ij}} M\,dx + N\,dy$$

$$= \sum_i \oint_{C_i} M\,dx + N\,dy + \sum_i \int_{\sum_{\substack{j \\ (j\neq i)}} L_{ij}} M\,dx + N\,dy$$

Note that, as C_i is not a common boundary of any other R_j,
$$\sum_i C_i = \partial R$$
$$\therefore \quad \sum_i \oint_{C_i} M\,dx + N\,dy = \oint_{\partial R} M\,dx + N\,dy$$

Finally, we have
$$L_{ji} = -L_{ij}$$
as R_i & R_j are located



on the two different sides of the common boundary.

$$\sum_i \int_{\substack{\sum_j L_{ij} \\ (j\neq i)}} Mdx+Ndy = \sum_i \sum_{\substack{j \\ (j\neq i)}} \int_{L_{ij}} Mdx+Ndy$$

$$= \sum_{\substack{i,j \\ (i\neq j)}} \int_{L_{ij}} Mdx+Ndy$$

$$= \sum_{i<j} \int_{L_{ij}} Mdx+Ndy + \sum_{\substack{\cancel{j}<\cancel{i} \\ i \quad j}} \int_{L_{\substack{\cancel{i}\cancel{j} \\ j\,i}}} Mdx+Ndy$$ (changing dummy indexes)

$$= \sum_{i<j} \left(\int_{L_{ij}} Mdx+Ndy + \int_{L_{ji}} Mdx+Ndy \right)$$

$$= \sum_{i<j} \left(\int_{L_{ij}} Mdx+Ndy + \int_{-L_{ij}} Mdx+Ndy \right)$$

$$= 0.$$

This 2^{nd} case basically include almost all situations in the level of Advanced Calculus.

The proof of <u>general</u> case needs "analysis" and will be omitted here. ※