(cont of eg 47)

$$\left(\begin{aligned} \vec{F} &= M\hat{i} + N\hat{j} + L\hat{k} \\ &= (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}. \end{aligned} \right)$$

To find f explicitly $\quad \vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \vec{F}$

$$\frac{\partial f}{\partial x} = y + e^z$$

$$\Rightarrow \quad f = \int (y+e^z)\,dx$$

$$= x(y+e^z) + \text{"const. in } x\text{"}$$

(function of y & z only)

$$f = x(y+e^z) + g(y,z) \quad \text{for some function } g(y,z)$$

Then take $\frac{\partial}{\partial y}$:

$$x+1 = \frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y}(y,z)$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 1$$

$$\Rightarrow \quad g = y + \text{const. in } y \text{ (const. in } x)$$

$$= y + h(z) \quad \text{for some function } h.$$

$$\Rightarrow \quad f = x(y+e^z) + y + h(z)$$

Then take $\frac{\partial}{\partial z}$

$$1+xe^z = \frac{\partial f}{\partial z} = xe^z + h'(z)$$

$$\Rightarrow \quad h'(z) = 1$$

$$\Rightarrow \quad h(z) = z + \text{const.} \leftarrow \text{real const.}$$

Hence $f = x(y+e^z) + y + z + C$, where C is a constant, is the required potential function. ※

(Note: This is equivalent to find f such that the total differential $df = M\,dx + N\,dy + L\,dz$)

Remark: To prove Thm 10 in $\mathbb{R}^2$, we need the Green's Thm (in $\mathbb{R}^3$, we need the Stokes' Thm)

Thm 11 (Green's Theorem)

Let $\Omega \subseteq \mathbb{R}^2$ be open, $\vec{F} = M\hat{i} + N\hat{j}$ be C^1 vector field on Ω;

C be a piecewise "smooth" simple closed anti-clockwise oriented curve enclosing a region R which lies entirely in Ω.

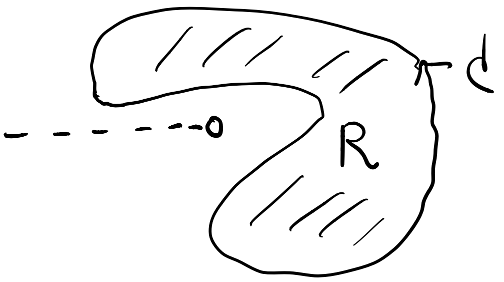
Then

- Normal Form $\oint_C \vec{F}\cdot\hat{n}\,ds = \boxed{\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy}$

- Tangential Form $\oint_C \vec{F}\cdot\hat{T}\,ds = \boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy}$

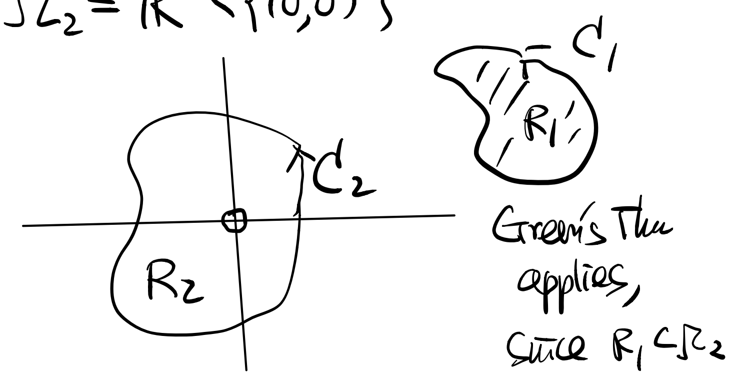
(Remark: The two forms are equivalent)

Note: $\Omega_1 = \mathbb{R}^2 \setminus \{x \le 0\}$



Green's Thm applies, since $R \subset \Omega_1$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$



$(0,0) \in R_2$, but $(0,0) \notin \Omega_2$

$\Rightarrow R_2 \not\subset \Omega_2$,

Green's Thm doesn't apply.

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eg 48 Verify both form of Green's Thm for

$$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j} \quad \text{on } \Omega = \mathbb{R}^2, \text{ is } C^\infty.$$

$$C = \text{unit circle}: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [0, 2\pi]$$

Then R = region enclosed by $C = \{x^2+y^2<1\}$ the unit disc.

(We also write $C = \partial R$ boundary of R)

Soln: $M = x-y$, $N = x$

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0.$$

On C, $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$

Normal form: $\quad LHS = \oint_C M\,dy - N\,dx$

$$= \int_0^{2\pi} [(\cos t - \sin t)\cos t - \cos t(-\sin t)]\,dt$$

$= \pi$ (check!)

$$\text{RHS} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1+0)\,dx\,dy = \pi$$

<u>Tangential form</u>:

$$\text{LHS} = \oint_C M\,dx + N\,dy = 2\pi \quad (\text{check!})$$

$$\text{RHS} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1-(-1))\,dx\,dy = 2\pi$$

(<u>Note</u>: This example shows that even the 2 forms are equivalent, but the <u>values</u> involved may differ.)

<u>Pf of Green's Thm</u> (tangential form)

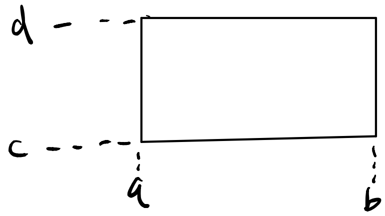
<u>Recall</u>: A region R is of special type:

type (1): If $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$ for some continuous functions $g_1(x)$ & $g_2(x)$.

type (2): If $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$ for some continuous functions $h_1(x)$ & $h_2(x)$.

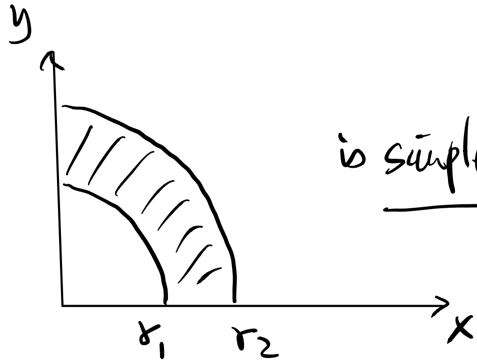
<u>Now</u>: If R is <u>both</u> type (1) and type (2), it said to be <u>simple</u>.

eg49: (i)



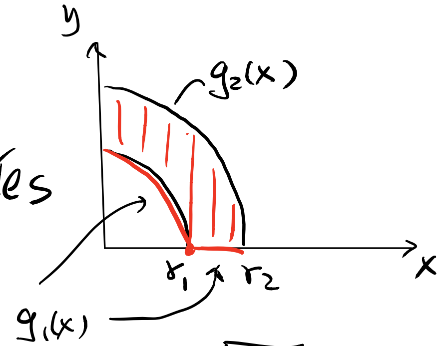
rectangle is simple

(ii)



is simple

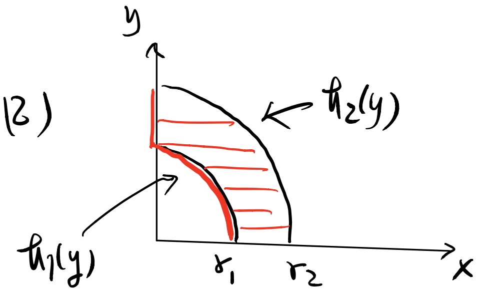
type(1): Yes



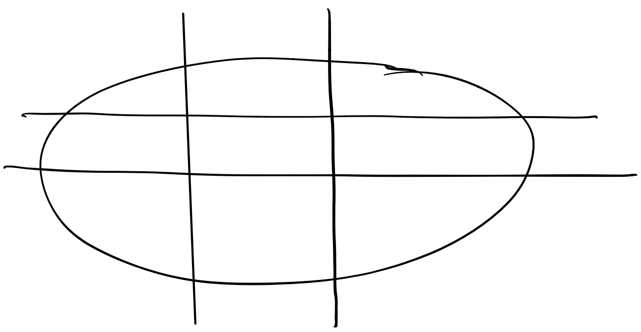
$$g_1(x)=\begin{cases}\sqrt{r_1^2-x^2}, & 0\le x\le r_1\\ 0, & r_1\le x\le r_2\end{cases}$$

continuous

type(2)



(iii)



2 intersections at most

2 intersections at most

$$\left.\begin{aligned}\forall a\in\mathbb{R}:\ &\#\{\partial R\cap\{x=a\}\}\le 2\\ &\#\{\partial R\cap\{y=a\}\}\le 2\end{aligned}\right\}\Rightarrow \underline{\text{simple}}$$

(provided ∂R is piecewise smooth)