If $p(x)$ and $q(x)$ are polynomials

$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0$ with $a_m > 0$ (i.e. $\deg p(x) = m$)

$q(x) = b_k x^k + a_{k-1} x^{k-1} + \cdots + b_1 x + b_0$ with $b_k > 0$ (i.e. $\deg q(x) = k$)

then

$$\lim_{n \to +\infty} \frac{p(n)}{q(n)} = \begin{cases} +\infty & \text{if } m > k \\ \dfrac{a_m}{b_k} & \text{if } m = k \\ 0 & \text{if } m < k \end{cases}$$

Following this idea :

e.g. Find $\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$

$$\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$$ roughly deg = 1

Following this idea:

e.g. Find $\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$

$$\lim_{n\to\infty} \frac{3n-1}{\sqrt{4n^2+2n}}$$ ← roughly deg = 1

$$= \lim_{n\to\infty} \frac{3-\frac{1}{n}}{\sqrt{4-\frac{2}{n}}}$$

Why?

$$\left(\lim_{n\to\infty} 4-\frac{2}{n} = 0 \implies \lim_{n\to\infty} \sqrt{4-\frac{2}{n}} = 0 \right)$$

Explain later!

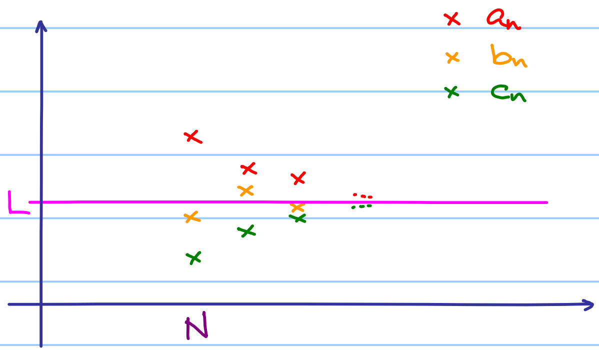
$$= \frac{3}{2}$$

Sandwich Theorem for Sequences

Let $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences of real numbers.

If $\exists N \in \mathbb{N}$ s.t. $a_n \leq b_n \leq c_n$ for $n \geq N$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$.

Geometrical meaning:



e.g. Find $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}}$

Note: $0 \leqslant \dfrac{1}{\sqrt{n+1}+\sqrt{n}} \leqslant \dfrac{1}{2\sqrt{n}} \quad \forall n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} 0 = \lim\limits_{n\to\infty} \dfrac{1}{2\sqrt{n}} = 0$

By sandwich theorem, $\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

e.g. Find $\lim_{n\to\infty} \frac{1}{n}\sin n$

Note: $-\frac{1}{n} \leq \frac{1}{n}\sin n \leq \frac{1}{n} \quad \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} -\frac{1}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$

By sandwich theorem, $\lim_{n\to\infty} \frac{1}{n}\sin n = 0$.

We may further generalize the above example as :

If $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded (i.e. $\exists M > 0$ s.t. $|b_n| \leq M \ \forall n \in \mathbb{N}$.) , then $\lim_{n\to\infty} a_n b_n = 0$.

proof :

FACT (without proof)

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0$$

Note : $-|a_n| \leq a_n \leq |a_n|$

$$-M \leq b_n \leq M$$

$$\therefore \ -M|a_n| \leq a_n b_n \leq M|a_n|$$

$$\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} -M|a_n| = \lim_{n\to\infty} M|a_n| = 0$$

By sandwich theorem , $\lim_{n\to\infty} a_n b_n = 0$.

Ex : Prove $\lim_{n\to\infty} \frac{(-1)^n}{n^2} = 0$

Hint : By using the above result.

Monotonic Sequence Theorem

Definition

Let $\{a_n\}$ be a sequence of real numbers.

(i) $\{a_n\}$ is said to be bounded above if $\exists M > 0$ s.t. $a_n < M$ — called an upper bound

(ii) $\{a_n\}$ is said to be bounded below if $\exists M > 0$ s.t. $a_n < M$ — called a lower bound

(iii) $\{a_n\}$ is said to be bounded if $\exists M > 0$ s.t. $|a_n| < M$ (i.e. $-M < a_n < M$)

bounded = both bounded above and below

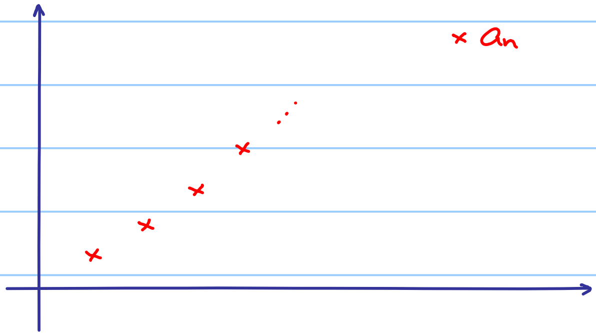
(iv) $\{a_n\}$ is said to be monotonic increasing if $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$

(v) $\{a_n\}$ is said to be monotonic decreasing if $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

Geometrical meaning :

$\{a_n\}$ is bounded above by M

But it may happen that a sharper bound M'



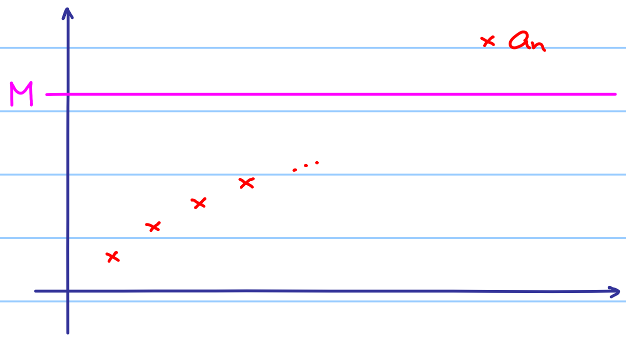
$\{a_n\}$ is monotonic increasing.

Combining together:

Monotonic Sequence Theorem:

If $\{a_n\}$ is bounded above (resp. below) and monotonic increasing (decreasing), then $\lim_{n\to\infty} a_n$ exists.

Geometrical meaning:



Caution:

$\{a_n\}$ is bounded above by M,

but $\lim_{n\to\infty} a_n$ is NOT necessary to be M.

e.g. Let $\{a_n\}$ be a sequence of real numbers defined by

$$a_1 = 1 \quad \text{and} \quad a_{n+1} = 1 + \frac{a_n}{1+a_n} \quad (n \geq 1)$$

Does $\lim_{n\to\infty} a_n$ exist?

i) Claim: $\{a_n\}$ is monotonic increasing

(Note: From the construction of the sequence, $a_n \geq 0 \ \forall n \in \mathbb{N}$)

Prove the statement "$a_{n+1} \geq a_n$" by induction:

Step 1: $a_2 - a_1 = \left(1 + \frac{a_1}{1+a_1}\right) - a_1 = \frac{3}{2} - 1 = \frac{1}{2} > 0$

Step 2: Assume $a_{k+1} \geq a_k$ for some $k \in \mathbb{N}$

$$
\begin{aligned}
a_{k+2} - a_{k+1} &= \left(1 + \frac{a_{k+1}}{1+a_{k+1}}\right) - \left(1 + \frac{a_k}{1+a_k}\right) \\
&= \frac{a_{k+1}}{1+a_{k+1}} - \frac{a_k}{1+a_k} \\
&= \frac{a_{k+1} - a_k}{(1+a_{k+1})(1+a_k)} \geq 0
\end{aligned}
$$

ii) $\{a_n\}$ is bounded above by 2.

$\therefore$ By monotonic sequence theorem, $\lim_{n\to\infty} a_n$ exists (But, what is the value?)

Let $\lim_{n\to\infty} a_n = A$

Note that $a_{n+1} = 1 + \frac{a_n}{1+a_n}$, taking limit on both sides

$$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} 1 + \frac{a_n}{1+a_n} = 1 + \frac{\lim_{n\to\infty} a_n}{1+\lim_{n\to\infty} a_n}$$

$$A = 1 + \frac{A}{1+A}$$

$$A^2 + A - 1 = 0$$

$$A = \frac{1+\sqrt{5}}{2} \text{ or } \frac{1-\sqrt{5}}{2} \text{ (rejected)}$$

Note : the limit is NOT 2 .

Constant e:

Consider a number $(1+\frac{1}{m})^n$ that depends on m and n and then

1) fix m, say $m=100$, n is getting larger and larger.

$n=10$ $\quad$ $n=100$ $\quad$ $n=1000$ $\quad \rightarrow +\infty$

$(1+\frac{1}{m})^n = 1.01^{10}$ $\quad$ $(1+\frac{1}{m})^n = 1.01^{100}$ $\quad$ $(1+\frac{1}{m})^n = 1.01^{1000}$ $\quad \rightarrow +\infty$

2) fix n, say $n=100$, m is getting larger and larger.

$m=10$ $\quad$ $m=100$ $\quad$ $m=1000$ $\quad \rightarrow +\infty$

$(1+\frac{1}{m})^n = 1.1^{100}$ $\quad$ $(1+\frac{1}{m})^n = 1.01^{100}$ $\quad$ $(1+\frac{1}{m})^n = 1.001^{100}$ $\quad \rightarrow 1$

How about setting $m=n$ and let them become larger and larger ?

$(1+\frac{1}{n})^n \to ?$ as $n \to +\infty$ (i.e. limit exists ? something between $+\infty$ and 1 ??)

$n=10$ $\quad (1+\frac{1}{n})^n = 1.1^{10} \approx 2.59374$

$n=100$ $\quad (1+\frac{1}{n})^n = 1.01^{100} \approx 2.70481$

$n=1000$ $\quad (1+\frac{1}{n})^n = 1.001^{1000} \approx 2.71692$

$\to +\infty$ $\quad \to 2.71828\cdots$

limit exists and call it e .

Idea of proof:

Let $a_n = (1+\frac{1}{n})^n$

1) Prove $\{a_n\}$ is monotonic increasing

2) Prove $\{a_n\}$ is bounded above by 3 .

Limits Involving e :

e.g. Find $\lim_{n\to\infty}(1+\frac{2}{n})^n$

$$\lim_{n\to\infty}(1+\frac{2}{n})^n = \lim_{n\to\infty}\left[(1+\frac{1}{(\frac{n}{2})})^{\frac{n}{2}}\right]^2$$

(let $y=\frac{n}{2}$, as $n\to\infty$, $y\to\infty$

$\lim_{n\to\infty}(1+\frac{1}{(\frac{n}{2})})^{\frac{n}{2}} = \lim_{y\to\infty}(1+\frac{1}{y})^y = e$)

$$= e^2$$

e.g. Find $\lim_{n \to \infty} \left(1 + \frac{1}{2n-1}\right)^n$

e.g. Find $\lim_{n\to\infty}\left(1+\frac{1}{2n-1}\right)^{n}$

$$\lim_{n\to\infty}\left(1+\frac{1}{2n-1}\right)^{n} = \lim_{n\to\infty}\left(1+\frac{1}{2n-1}\right)^{\frac{1}{2}(2n-1)+\frac{1}{2}}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{2n-1}\right)^{2n-1}\right]^{\frac{1}{2}} \cdot \left(1+\frac{1}{2n-1}\right)^{\frac{1}{2}}$$

$$= e^{\frac{1}{2}} \cdot 1$$

$$= e^{\frac{1}{2}}$$

e.g. Find $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}$

e.g. Find $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n}$

$$\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^{-n} = \lim_{n\to\infty}\left(\frac{n-1}{n}\right)^{-n}$$

$$= \lim_{n\to\infty}\left(\frac{n}{n-1}\right)^{n}$$

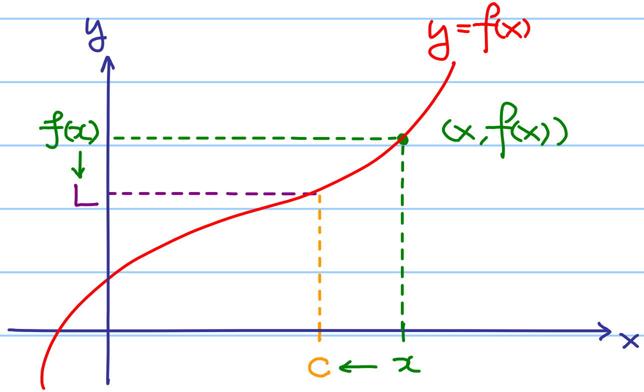
$$= \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^{n-1}\cdot\left(1+\frac{1}{n-1}\right)$$

$$= e\cdot 1 = e$$

Limits of Functions :

Limit of a function :

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer[+] to c from both sides, then L is called the limit of $f(x)$ at c. We write $\lim\limits_{x \to c} f(x) = L$



+ Note : a little bit misleading !

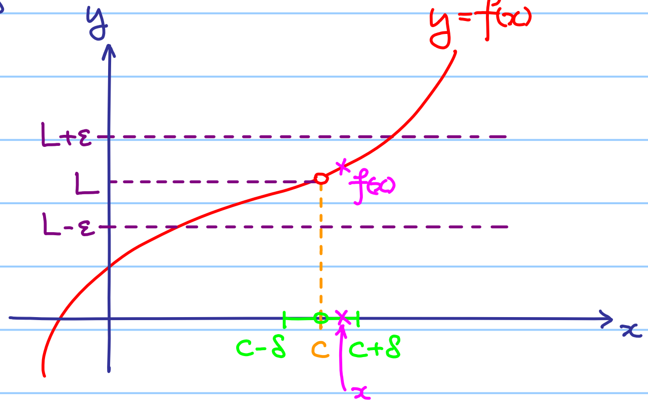
$f(c)$ may NOT equal to L, even it may be undefined !

Definition

Let $f:\mathbb{R}\to\mathbb{R}$ be a function and $c\in\mathbb{R}$.

$L\in\mathbb{R}$ is said to be the limit of f at the point c if

$\forall\, \varepsilon>0$, $\exists\, \delta>0$ s.t. $|f(x)-L|<\varepsilon$ $\forall\, 0<|x-c|<\delta$



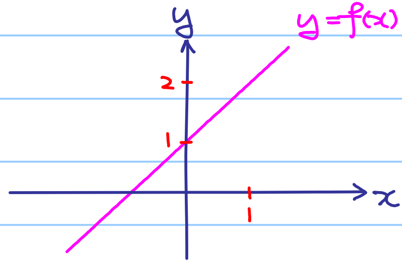
Meaning: No matter how small ε you give me,

i.e. $x\neq c$

I can always find $\delta>0$ s.t. if x is a point with $0<\mathrm{dist}(x,c)<\delta$

then $f(x)$ lies in the ε-tunnel (ε-neighborhood of L)

e.g. If $f(x) = x+1$, find $\lim_{x \to 1} f(x)$



+ x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1

$f(x)$ tends to 2 as x tends to 1.

We write $\lim_{x \to 1} f(x) = 2$.

Remarks:

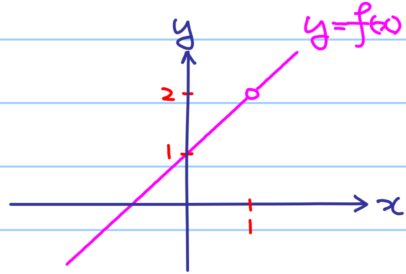
1) + The table only gives an intuitive idea, but NOT a rigorous proof!

2) Do NOT regard as putting $x = 1$ into $f(x)$ and get $f(1) = 2$!

e.g. Let $f : \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{x^2 - 1}{x - 1}$, $\quad x \neq 1$.

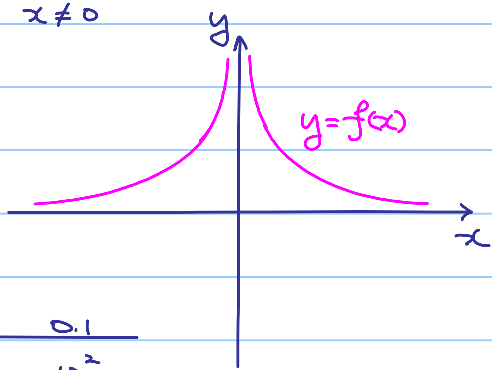
We can rewrite f as the following:

$$f(x) = \begin{cases} x + 1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2.001	2.01	2.1

e.g. Let $f: \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$, $x \neq 0$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	undefined	10^6	10^4	10^2

$f(x)$ tends to $+\infty$ (NOT a real number) as x tends to 0

$\therefore \lim_{x\to 0} f(x)$ does NOT exist.

(But, some still write $\lim_{x\to 0} f(x) = +\infty$.)

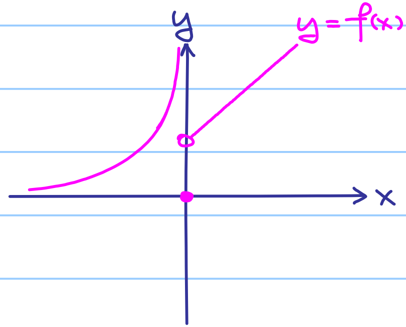
Right Hand Limit and Left Hand Limit :

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer to c from right (resp. left) hand side, then L is called the right (resp. left) hand limit of $f(x)$ at c.

We write $\lim_{x \to c^+} f(x) = L$ (resp. $\lim_{x \to c^-} f(x) = L$)

e.g.

If $f(x) = \begin{cases} x+1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x < 0 \end{cases}$



$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x+1 = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{1}{x^2}$$ (does NOT exist)

$$f(0) = 0$$

Remark :

Right hand limit and left hand limit of a function at a point is NOT necessary to be the same !

FACT :

$\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c^+} f(x) = \lim_{x \to c^-} f(x) = L$

(i.e. both right and left hand limit exist and equal to L.)

FACT (Without proof)

(1) If k is a constant, $\lim_{x \to c} k = k$

regarded as constant function $f(x) = k$

(2) $\lim_{x \to c} x = c$

Algebraic Properties of Limits :

If $\lim\limits_{x \to c} f(x)$ and $\lim\limits_{x \to c} g(x)$ exist (very important!), then

(1) $\lim\limits_{x \to c} (f(x) + g(x)) = \lim\limits_{x \to c} f(x) + \lim\limits_{x \to c} g(x)$

(2) $\lim\limits_{x \to c} (f(x) - g(x)) = \lim\limits_{x \to c} f(x) - \lim\limits_{x \to c} g(x)$

(3) $\lim\limits_{x \to c} (f(x) g(x)) = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x)$

(4) $\lim\limits_{x \to c} \left(\dfrac{f(x)}{g(x)}\right) = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)}$ if $\lim\limits_{x \to c} g(x) \neq 0$

e.g. Find $\lim_{x \to 2} 3x^2 - 5$

① $\lim_{x \to 2} x = 2$, so $\lim_{x \to 2} x^2 = \lim_{x \to 2} (x \cdot x) \overset{\text{By (3)}}{=} \lim_{x \to 2} x \cdot \lim_{x \to 2} x$

$= 2 \cdot 2 = 4$

② $\lim_{x \to 2} 3 = 3$, $\lim_{x \to 2} x^2 = 4$, so $\lim_{x \to 2} 3x^2 \overset{\text{By (3)}}{=} \lim_{x \to 2} 3 \cdot \lim_{x \to 2} x^2$

$= 3 \cdot 4 = 12$

③ $\lim_{x \to 2} 3x^2 = 12$, $\lim_{x \to 2} 5 = 5$, so $\lim_{x \to 2} 3x^2 - 5 \overset{\text{By (2)}}{=} \lim_{x \to 2} 3x^2 - \lim_{x \to 2} 5$

$= 12 - 5 = 7$

But what we write :

$$\lim_{x \to 2} 3x^2 - 5 = 3\left(\lim_{x \to 2} x\right)^2 - 5 = 7$$

e.g. Find $\lim\limits_{x \to 1} \dfrac{3x^2 - 8}{x - 2}$

$$\lim_{x \to 1} \frac{3x^2 - 8}{x - 2} = \frac{3\left(\lim\limits_{x \to 1} x\right)^2 - 8}{\left(\lim\limits_{x \to 1} x\right) - 2} = \frac{3(1) - 8}{1 - 2} = 5$$

e.g. Think:

$$\lim_{x\to 0} \frac{1}{x} = \lim_{x\to 0} x \cdot \frac{1}{x^2} = \underbrace{\lim_{x\to 0} x}_{=\,0} \cdot \lim_{x\to 0} \frac{1}{x^2} \underset{(*)}{=} 0$$

But we know $\lim_{x\to 0} \frac{1}{x}$ does NOT exist.

What's wrong?

Ans: $\lim_{x\to 0} \frac{1}{x^2}$ does NOT exist, so we cannot use (3) at (*).

e.g. Find $\lim\limits_{x \to 1} \dfrac{x^2-1}{x^2-3x+2}$

Note $\lim\limits_{x \to 1} x^2-3x+2 = 0$, so we cannot use (4).

$$\lim_{x \to 1} \frac{x^2-1}{x^2-3x+2} = \lim_{x \to 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x \to 1} \frac{x+1}{x-2} \overset{\text{By (4)}}{=} \frac{\lim\limits_{x \to 1} x+1}{\lim\limits_{x \to 1} x-2} = \frac{2}{-1} = -2$$

e.g. Let $f: \mathbb{R} \setminus \{1\} \longrightarrow \mathbb{R}$ defined by $f(x) = \dfrac{\sqrt{x}-1}{x-1}$, $x \neq 1$.

Find $\lim\limits_{x \to 1} f(x)$.

Note : For $x \neq 1$ ($x - 1 \neq 0$, denominator is nonzero.)

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{1}{\sqrt{x}+1}$$

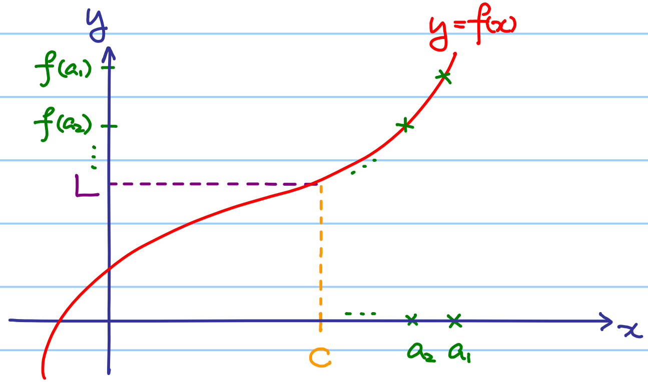
$$\lim_{x \to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \to 1} \frac{1}{\sqrt{x}+1}$$ (We only concern those x near 1 but NOT equal to 1)

$$= \frac{1}{2}$$ (Still the same, do NOT regard as putting $x = 1$)

Relation Between Limits of Sequences and Functions

Theorem :

$\lim\limits_{x \to c} f(x) = L \iff \forall$ sequence $\{a_n\}$ with $\lim\limits_{n\to\infty} a_n = c$, we have $\lim\limits_{n\to\infty} f(a_n) = L$



Theorem :

$\lim_{x\to c} f(x) = L \iff \forall$ sequence $\{a_n\}$ with $\lim_{n\to\infty} a_n = c$, we have $\lim_{n\to\infty} f(a_n) = L$

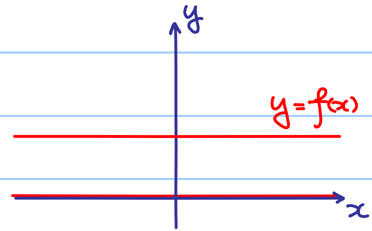
In fact, if we want to show $\lim_{x\to c} f(x) = L$, it is quite impossible to check infinitely many sequence. This statement is useful in reverse direction :

1) If $\exists$ $\{a_n\}$ s.t. $\lim_{n\to\infty} a_n = c$, but $\lim_{n\to\infty} f(a_n)$ does NOT exist,
then $\lim_{x\to c} f(x)$ does NOT exist.

2) If $\exists$ $\{a_n\}, \{b_n\}$ s.t. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = c$, but $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$
then $\lim_{x\to c} f(x)$ does NOT exist.

e.g. Let $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$



It seems the graph consists of two straight lines, but in fact infinitely many holes are there.

Consider sequences $\{a_n\}$, $\{b_n\}$ defined by

$$a_n = \frac{1}{n} \in \mathbb{Q} \quad , \quad b_n = \frac{\sqrt{2}}{n} \in \mathbb{R} \setminus \mathbb{Q}$$

Then $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = 0$, but $\lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} 1 = 1$

$\lim\limits_{n\to\infty} f(b_n) = \lim\limits_{n\to\infty} 0 = 0$

$\therefore \lim\limits_{x\to 0} f(x)$ does NOT exist.

In fact, with little modification, we can show $\lim\limits_{x\to c} f(x)$ does NOT exist $\forall\, c \in \mathbb{R}$.

Ex: Let $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Show that $\lim_{x \to 0} f(x)$ does NOT exist.

Hint: Consider sequences $\{a_n\}$, $\{b_n\}$ defined by

$$a_n = \frac{1}{2n\pi}, \quad b_n = \frac{1}{(2n+\frac{1}{2})\pi}$$

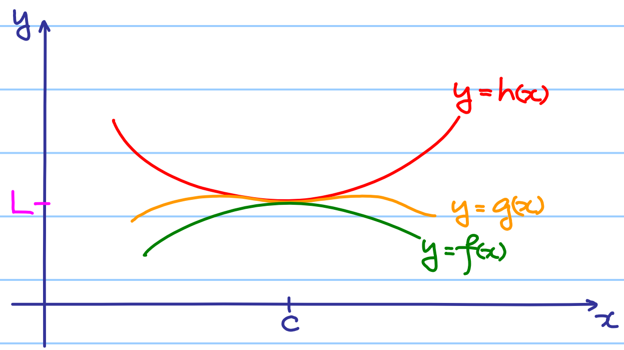
Sandwich Theorem for Functions

Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions and $c \in \mathbb{R}$

If $f(x) \le g(x) \le h(x)$ for all x in a neighborhood of c and $\lim_{x\to c} f(x) = \lim_{x\to c} h(x) = L$,

then $\lim_{x\to c} g(x) = L$

Geometrical meaning :



e.g. Prove that $\lim_{x \to 0} x^2 \cos^2 \frac{1}{x} = 0$

Note that $0 \le x^2 \cos^2 \frac{1}{x} \le x^2$ and $\lim_{x \to 0} 0 = \lim_{x \to 0} x^2 = 0$

By sandwich theorem, $\lim_{x \to 0} x^2 \cos^2 \frac{1}{x} = 0$.

FACT (without proof)

$$\lim_{x\to c} f(x) = 0 \iff \lim_{x\to c} |f(x)| = 0$$

e.g. $\lim_{x\to 0} x = 0 \Rightarrow \lim_{x\to 0} |x| = 0$

e.g. Prove that $\lim_{x\to 0} x\cos\frac{1}{x} = 0$

Note that $-|x| \leq x\cos\frac{1}{x} \leq |x|$ and $\lim_{x\to 0} -|x| = \lim_{x\to 0} |x| = 0$

By sandwich theorem, $\lim_{x\to 0} x\cos\frac{1}{x} = 0$.