Review :

1) Notations :

Set : collection of objects (elements)

$\subseteq$: subset

$\in$: belongs to

e.g. $S = \{1, 2, 3\}$

That means S is a set containing 3 elements, namely 1, 2 and 3.

OR : $1, 2, 3 \in S$

If $T = \{1, 2, 3, 4\}$, then we say S is a subset of T, or $S \subseteq T$.

That means all elements in S and also in T.

Notations often used in this course :

$\mathbb{N}$: set of all natural numbers

$\mathbb{Q}$: set of all rational numbers

$\mathbb{R}$: set of all real numbers

$[a,b]$: set of all real numbers x such that $a \leq x \leq b$

(a,b) : set of all real numbers x such that $a < x < b$

$[a,+\infty)$: set of all real numbers x such that $a \leq x$

$\mathbb{R}\setminus\{a\}$: set of all real numbers except the number a

e.g. Set of all positive even integers

$= \{2, 4, 6, \dots\}$

$= \{2m \mid m \in \mathbb{N}\}$

i.e. this set consists of elements of the form $2m$ such that $m \in \mathbb{N}$.

Ex: Set of all positive odd integers $=$? (How to describe?)

Ans: $= \{2m-1 \mid m \in \mathbb{N}\}$

e.g. Set of all real numbers x such that $a \leq x \leq b$

$= \{x \in \mathbb{R} \mid a \leq x \leq b\}$

$\forall$: for all

$\exists$: there exists

$\exists!$: there exists unique

$\Rightarrow$: implies

$\Leftrightarrow$: if and only if (equivalent to)

s.t. : such that

e.g. $\forall\, y \in (0,+\infty),\ \exists\, x \in \mathbb{R}$ s.t. $x^2 = y$.

$\downarrow$ translate

For all positive real number y, there exists (at least one) real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ or $-\sqrt{y}$)

e.g. $\forall\, y \in (0,+\infty),\ \exists!\, x \in (0,+\infty)$ s.t. $x^2 = y$.

$\downarrow$ translate

For all positive real number y, there exists unique positive real number x such that $x^2 = y$.

(In fact, $x = \sqrt{y}$ only)

e.g. If $x > 0$, $y = \sqrt{x} \Rightarrow y^2 = x$

but $y^2 = x \not\Rightarrow y = \sqrt{x}$ (Why?)

e.g. In a ΔABC,

$\angle ABC = 90^\circ \Rightarrow AB^2 + BC^2 = AC^2$ (Pyth. thm.)

$AB^2 + BC^2 = AC^2 \Rightarrow \angle ABC = 90^\circ$ (Converse of Pyth. thm.)

If both statements are true, we say

$\angle ABC = 90^\circ$ if and only if $AB^2 + BC^2 = AC^2$

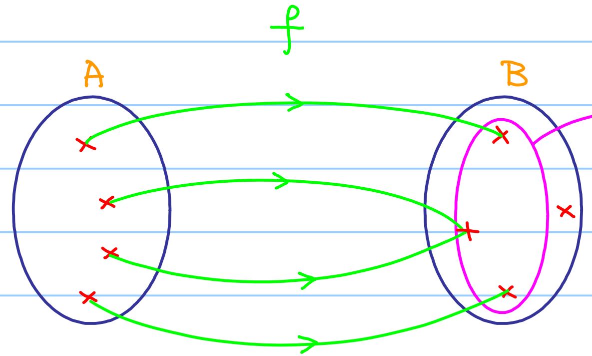
and denote it by $\angle ABC = 90^\circ \Leftrightarrow AB^2 + BC^2 = AC^2$

2) Functions :

Function : A function is a rule that assigns to each object in a set A exactly one object in a set B.

set A : domain (input)

set B : range (output)



$\text{image}(f) \subseteq B$: image of f

A function f from A to B

We denote it by $f : A \to B$

$\text{image}(f) = f(A) := \{ f(x) \in B \mid x \in A \}$

$:=$ defined by

e.g. If

1) $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$ image $(f) = [0, +\infty)$

2) $f: [-1, 2) \to \mathbb{R}$ defined by $f(x) = x^2$ image $(f) = [0, 4)$

e.g. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2 + 4$

$f(-3) = (-3)^2 + 4 = 13$

input output

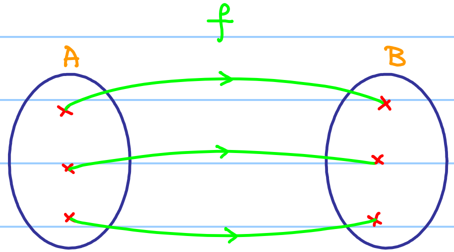
OR write : $y = x^2 + 4$

dependent variable

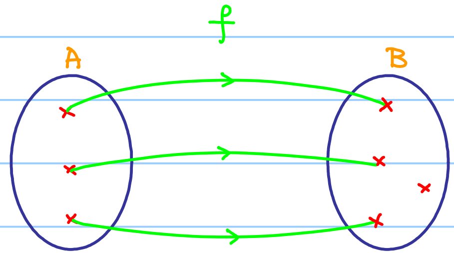
independent variable

Injective and Surjective Functions

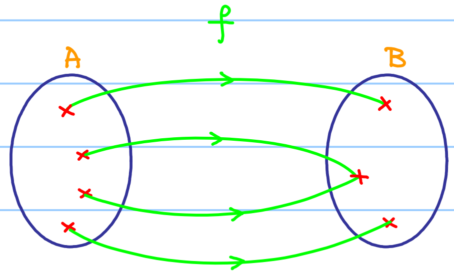
Intuitive idea :

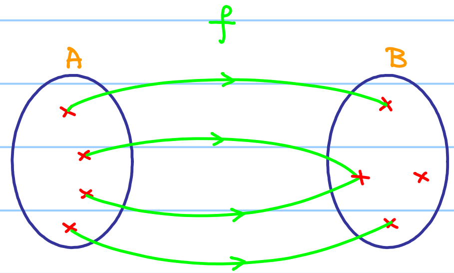


injective + surjective



injective but NOT surjective





surjective but NOT injective

injective : every $y \in \text{image}(f)$ comes from exactly one $x \in A$

surjective : every $y \in B$ comes from one $x \in A$

Definition :

Let $f: A \to B$ be a function.

1) f is said to be an injective function if

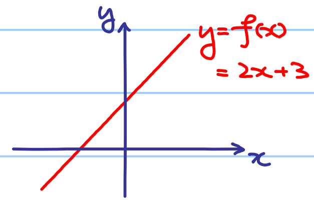
$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

2) f is said to be a surjective function if

$$\forall y \in B, \exists x \in A \text{ s.t. } f(x) = y \qquad (f(A) = B)$$

If a function is both injective and surjective, then it is said to be a bijective function.

e.g. Show $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x + 3$ is a bijective function



e.g. Show $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x+3$ is a bijective function

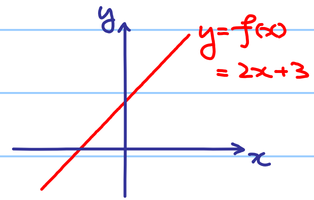
1) Injective :

$$f(x_1) = f(x_2)$$

$$\Rightarrow \quad 2x_1 + 3 = 2x_2 + 3$$

$$\Rightarrow \quad x_1 = x_2$$

$\therefore$ f is injective.



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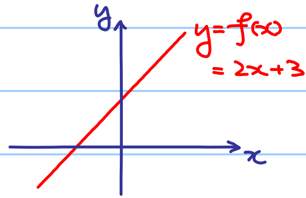


2) surjective :

Let $y \in \mathbb{R}$,

take $x = \dfrac{y-3}{2} \in \mathbb{R}$

then $f(x) = f\left(\dfrac{y-3}{2}\right)$

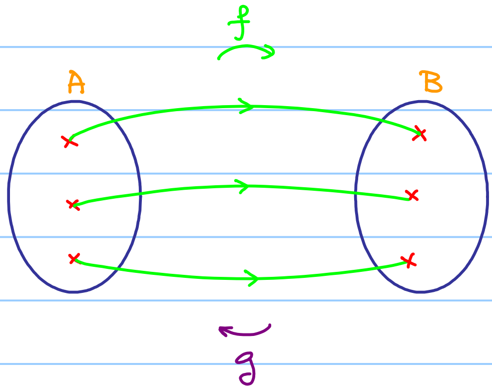
$= 2\left(\dfrac{y-3}{2}\right) + 3$

$= y$

$\therefore$ f is surjective .

Inverse of a Function

Intuitive idea :



Definition :

Let $f: A \to B$ be a function. If $g: B \to A$ is a function such that

1) $g(f(x)) = x \quad \forall x \in A$

2) $f(g(y)) = y \quad \forall y \in B$

Then g is said to be an inverse of f.

FACT : 1) Once an inverse of f exists, it is unique, we denote it by f^{-1}.

2) f has an inverse $\Leftrightarrow$ f is bijective.

e.g.

	injective	surjective
$f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$	✗	✗
$f: \mathbb{R} \to [-1, 1]$ defined by $f(x) = \sin x$	✗	✓
$f: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ defined by $f(x) = \sin x$	✓	✓

$\therefore$ We can define arcsin function !

$\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$

We write $\sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$, then

$\sin^{-1}(\sin x) = x \quad \forall x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$\sin(\sin^{-1} y) = y \quad \forall y \in [-1, 1]$

Sequences of Real Numbers

e.g. $a_1 = 2$, $a_2 = \pi$, $a_3 = 1$, $\cdots$

OR write as $\{2, \pi, 1, \cdots\}$ (No pattern)

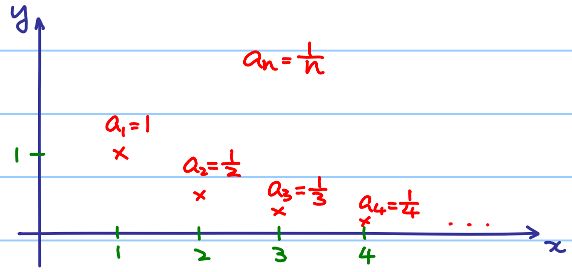
e.g. Sequences having patterns.

$a_1 = 1$, $a_2 = 2$, $a_3 = 4$, $\cdots$ in general, $a_n = 2^{n-1}$

$a_1 = 1$, $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $\cdots$ in general, $a_n = \frac{1}{n}$

$a_1 = -1$, $a_2 = 1$, $a_3 = -1$, $\cdots$ in general, $a_n = (-1)^n$

Remark : A sequence of real numbers can be regarded as a function $f: \mathbb{N} \to \mathbb{R}$
and $a_n = f(n)$ (i.e. given $n \in \mathbb{N}$, return the n-th entry of the sequence)
A sequence can be understood by the following diagram.



Any observation?

When n is getting larger and larger, a_n is getting closer and closer to 0.

Limits of Sequences

Informal definition :

Let a_n be a sequence of real numbers.

If n is getting larger and larger, a_n is getting closer and closer to $L \in \mathbb{R}$, then we say L is the limit of the sequence a_n and we denote it by

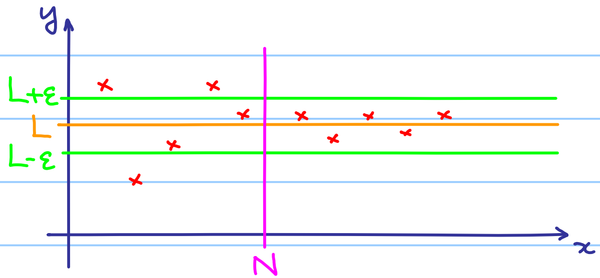
$$\lim_{n \to \infty} a_n = L .$$

Definition :

Let $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$ and $L \in \mathbb{R}$.

L is said to be the limit of the sequence a_n if

$\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n - L| < \varepsilon$ $\forall n \geq N$.



Meaning : No matter how small ε you give me,

I can always find a $N \in \mathbb{N}$ s.t. the tail (a_n with $n \geq N$) of sequence

lies in the ε-tunnel (ε-neighborhood of L)

e.g. $\lim_{n\to\infty} \frac{1}{n} = 0$.

$\lim_{n\to\infty} (-1)^n$ does NOT exist.

$\lim_{n\to\infty} 2^{n-1}$ does NOT exist.

FACT (without proof)

1) If $a_n = k \ \forall n \in \mathbb{N}$ (constant sequence), then $\lim_{n\to\infty} a_n = k$.

2) If $k>0$ and $a_n = n^{-k} = \frac{1}{n^k}$, then $\lim_{n\to\infty} a_n = 0$.

Algebraic Properties of Limits

If $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = M$ (very important) , then

1) $\lim_{n\to\infty} a_n + b_n = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = L + M$

2) $\lim_{n\to\infty} a_n - b_n = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = L - M$

3) $\lim_{n\to\infty} a_n b_n = (\lim_{n\to\infty} a_n)(\lim_{n\to\infty} b_n) = LM$

4) If $M \neq 0$, $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{L}{M}$.

e.g. Find $\lim_{n \to \infty} \frac{2}{n} + 3$

e.g. Find $\lim_{n\to\infty} \frac{2}{n} + 3$

① $\lim_{n\to\infty} 2 = 2$, $\lim_{n\to\infty} \frac{1}{n} = 0$ $\overset{\text{by 3}}{\Longrightarrow}$ $\lim_{n\to\infty} \frac{2}{n} = (\lim_{n\to\infty} 2)(\lim_{n\to\infty} \frac{1}{n}) = 2 \cdot 0 = 0$

② $\lim_{n\to\infty} \frac{2}{n} = 0$, $\lim_{n\to\infty} 3 = 3$ $\overset{\text{by 1}}{\Longrightarrow}$ $\lim_{n\to\infty} \frac{2}{n} + 3 = \lim_{n\to\infty} \frac{2}{n} + \lim_{n\to\infty} 3 = 0 + 3 = 3$

e.g. Find $\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$

$\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$ (We cannot use 4 , why ?)

e.g. Find $\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$

$$\lim_{n\to\infty} \frac{n^2+3}{2n^2-4n}$$ (We cannot use 4, why?)

$$= \lim_{n\to\infty} \frac{1+\frac{3}{n^2}}{2-\frac{4}{n}}$$ (Now, we can use 4!)

$$= \frac{\lim_{n\to\infty} 1+\frac{3}{n^2}}{\lim_{n\to\infty} 2-\frac{4}{n}} = \frac{1}{2}$$

Ex: Find $\lim_{n\to\infty} \frac{3n+1}{n^2-2n}$, $\lim_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ (if exist)

Ans: $\lim_{n\to\infty} \frac{3n+1}{n^2-2n} = 0$, $\lim_{n\to\infty} \frac{n^3+2n}{2n^2+1}$ does NOT exist.

Any observation ?

e.g. Find $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$$

e.g. Find $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n}$$

$$= \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n}\right) \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$= 0$$

why it is true ?
explain later !