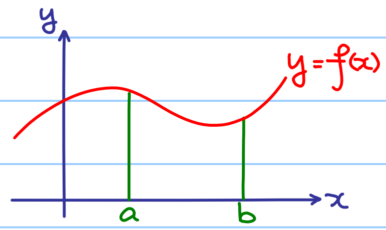
Inequalities Involving Integrals

Theorem : If $f:[a,b]\to\mathbb{R}$ is a continuous function such that $f(x)\geq 0$ for all $x\in[a,b]$, then $\int_a^b f(x)\,dx \geq 0$.



Corollary : If $f, g : [a,b] \to \mathbb{R}$ are continuous functions such that $f(x) \geq g(x)$ for all $x \in [a,b]$, then
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

proof : $f(x) - g(x) \geq 0$ for all $x \in [a,b]$

$$\Rightarrow \int_a^b f(x) - g(x)\,dx \geq 0$$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

Corollary: If $f:[a,b]\to\mathbb{R}$ is a continuous function such that

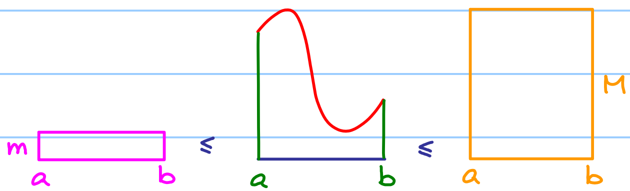
$m \le f(x) \le M$ for all $x\in[a,b]$, where $m, M \in \mathbb{R}$, then

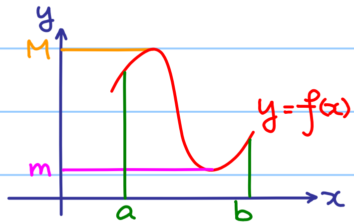
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

proof: $m \le f(x) \le M$ for all $x \in [a,b]$

$$\Rightarrow \int_a^b m\,dx \le \int_a^b f(x)\,dx \le \int_a^b M\,dx$$

$$\Rightarrow m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$





e.g. Define $a_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$ for $n \in \mathbb{N}$, show that $\frac{1}{2(n+1)} \leq a_n \leq \frac{1}{n+1}$.

Hence, find $\lim_{n\to\infty} a_n$.

e.g. Define $a_n = \int_0^1 \frac{x^n}{1+x^2}\,dx$ for $n \in \mathbb{N}$, show that $\frac{1}{2(n+1)} \leq a_n \leq \frac{1}{n+1}$.

Hence, find $\lim_{n\to\infty} a_n$.

Note: For $0 \leq x \leq 1$,

$$1 \leq 1+x^2 \leq 2$$

$$\frac{1}{2} \leq \frac{1}{1+x^2} \leq 1$$

$$\frac{x^n}{2} \leq \frac{x^n}{1+x^2} \leq x^n$$

$$\int_0^1 \frac{x^n}{2}\,dx \leq \int_0^1 \frac{x^n}{1+x^2}\,dx \leq \int_0^1 x^n\,dx$$

$$\frac{1}{2(n+1)} \leq \int_0^1 \frac{x^n}{1+x^2}\,dx \leq \frac{1}{n+1}$$

$\because \lim_{n\to\infty} \frac{1}{2(n+1)} = \lim_{n\to\infty} \frac{1}{n+1} = 0$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty} \int_0^1 \frac{x^n}{1+x^2}\,dx = 0$

e.g. Let $I_n = \int_0^1 e^t t^n \, dt$ where n is nonnegative integer.

a) Prove that $I_n = e - nI_{n-1}$ for $n \geq 1$.

Hence, deduce that $I_n = (-1)^{n+1} n! + e \sum_{r=0}^{n} (-1)^i \frac{n!}{(n-r)!}$.

b) Show that $\frac{1}{n+1} \leq I_n \leq \int_0^1 e t^n \, dt < \frac{e}{n}$ for all $n \geq 1$.

c) Hence, prove that e is an irrational number.

a) For $n \geq 1$, $I_n = \int_0^1 e^t t^n \, dt$

$= \int_0^1 t^n \, de^t$

$= [t^n e^t]_0^1 - \int_0^1 e^t \, dt^n$

$= e - \int_0^1 n e^t t^{n-1} \, dt$

$= e - n I_{n-1}$

$$
\begin{aligned}
I_n &= e - nI_{n-1} \\
&= e - n\left[e - (n-1)I_{n-2}\right] \\
&= e - ne + n(n-1)I_{n-2} \\
&= e - ne + n(n-1)\left[e - (n-2)I_{n-3}\right] \\
&= e - ne + n(n-1)e - n(n-1)(n-2)I_{n-3} \\
&\ \ \vdots \\
&= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2 + (-1)^{n}n(n-1)(n-2)\cdots 2\cdot 1\cdot I_0 \\
&= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2 + (-1)^{n}n(n-1)(n-2)\cdots 2\cdot 1\cdot (e-1) \\
&= e - ne + n(n-1)e - \cdots + (-1)^{n-1}n(n-1)(n-2)\cdots 2 + (-1)^{n}n(n-1)(n-2)\cdots 2\cdot 1\cdot e \\
&\quad + (-1)^{n+1}n(n-1)(n-2)\cdots 2\cdot 1 \\
&= (-1)^{n+1}n! + e\sum_{r=0}^{n}(-1)^{i}\frac{n!}{(n-r)!}
\end{aligned}
$$

$$
\begin{aligned}
I_0 &= \int_0^1 e^t\,dt \\
&= \left[e^t\right]_0^1 \\
&= e - 1
\end{aligned}
$$

b) Note: For $0 \leq t \leq 1$, $t^n \leq e^t t^n \leq e t^n$

$$\int_0^1 t^n \, dt \leq \underbrace{\int_0^1 e^t t^n \, dt}_{= I_n} \leq \int_0^1 e t^n \, dt$$

$$\int_0^1 t^n \, dt = \left[\frac{1}{n+1} t^{n+1}\right]_0^1 = \frac{1}{n+1}$$

$$\int_0^1 e t^n \, dt = e \int_0^1 t^n \, dt = \frac{e}{n+1} < \frac{e}{n}$$

$$\therefore \ \frac{1}{n+1} \leq I_n \leq \int_0^1 e t^n \, dt < \frac{e}{n}$$

c) From (a) and (b),

$$\frac{1}{n+1} \leq I_n < \frac{e}{n}$$

$$\frac{1}{n+1} \leq (-1)^{n+1} n! + e \sum_{r=0}^{n} (-1)^i \frac{n!}{(n-r)!} < \frac{e}{n}$$

$$\frac{1}{e(n+1)} \leq \frac{(-1)^{n+1} n!}{e} + \sum_{r=0}^{n} (-1)^i \frac{n!}{(n-r)!} < \frac{1}{n}$$

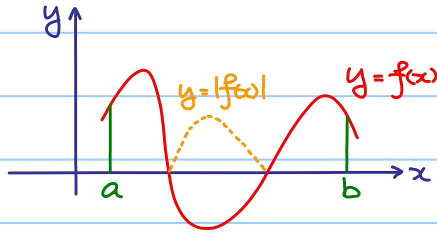
If e is rational, when we consider a sufficiently large n,

then $\frac{(-1)^{n+1} n!}{e} + \sum_{r=0}^{n} (-1)^i \frac{n!}{(n-r)!}$ is an integer which is impossible as $0 < \frac{1}{e(n+1)}$ and $\frac{1}{n} < 1$!

$\therefore$ e is irrational.

Corollary : If $f:[a,b]\to\mathbb{R}$ is a continuous function, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$



$$\left(\text{i.e. } -\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx \right)$$

proof : Note : $-|f(x)| \le f(x) \le |f(x)|$

$$-\int_a^b |f(x)|\,dx \le \int_a^b f(x)\,dx \le \int_a^b |f(x)|\,dx$$

e.g. Let $I_n = \frac{1}{n}\int_0^1 \frac{\sin nx}{1+x^2}\,dx$, for $n \in \mathbb{N}$, prove that $|I_n| \leq \frac{\pi}{4n}$.

Hence, deduce $\lim_{n\to\infty} I_n = 0$.

e.g. Let $I_n = \frac{1}{n}\int_0^1 \frac{\sin nx}{1+x^2}\,dx$, for $n \in \mathbb{N}$, prove that $|I_n| \leq \frac{\pi}{4n}$.

Hence, deduce $\lim_{n\to\infty} I_n = 0$.

$$|I_n| = \frac{1}{n}\left|\int_0^1 \frac{\sin nx}{1+x^2}\,dx\right|$$

$$\leq \frac{1}{n}\int_0^1 \left|\frac{\sin nx}{1+x^2}\right| dx$$

$$\leq \frac{1}{n}\int_0^1 \frac{1}{1+x^2}\,dx$$

$$= \frac{1}{n}\left[\tan^{-1}x\right]_0^1$$

$$= \frac{\pi}{4n}$$

$0 \leq |I_n| \leq \frac{\pi}{4n}$ and $\lim_{n\to\infty} 0 = \lim_{n\to\infty} \frac{\pi}{4n} = 0$

$\therefore$ By sandwich theorem, $\lim_{n\to\infty} |I_n| = 0$ and so $\lim_{n\to\infty} I_n = 0$.