e.g. If $f(x)$ is differentiable, then $k \cdot f(x)$ is differentiable.

and $(kf)'(x) = k \cdot f'(x)$ (or write $\frac{d}{dx} kf(x) = k\frac{df}{dx}$).

Idea: Let $g(x) = k$, then $g'(x) = 0$.

Apply product rule, the result follows.

e.g. Find $\frac{d}{dx}(3x^2+7x-2)$

$$\frac{d}{dx}(3x^2+7x-2) = \frac{d}{dx}(3x^2) + \frac{d}{dx}(7x) - \frac{d}{dx}(2)$$ Apply ① and ②

$$= 3\frac{d}{dx}(x^2) + 7\frac{d}{dx}(x) - \frac{d}{dx}(2)$$
$$= 3(2x) + 7(1) - 0$$
$$= 6x + 7$$

e.g. Find the derivative of the function $(3x^2-5x+1)(2x+7)$

$$\frac{d}{dx}[(3x^2-5x+1)(2x+7)]$$
$$= \left[\frac{d}{dx}(3x^2-5x+1)\right](2x+7) + (3x^2-5x+1)\left[\frac{d}{dx}(2x+7)\right]$$ Apply ③ product rule
$$= (6x-5)(2x+7) + (3x^2-5x+1)(2)$$
$$= 18x^2+22x-33$$

Ex: Try to compare: Expand $(3x^2-5x+1)(2x+7)$ and get $6x^3+11x^2-33x+7$

Then differentiate, get the same result?

e.g. Find the derivative of the function $\frac{2x}{x^2+1}$.

$$\frac{d}{dx}\frac{2x}{x^2+1} = \frac{\left[\frac{d}{dx}(2x)\right](x^2+1) - (2x)\left[\frac{d}{dx}(x^2+1)\right]}{(x^2+1)^2}$$

$$= \frac{2(x^2+1) - 2x(2x)}{(x^2+1)^2}$$

$$= \frac{-2x^2+2}{(x^2+1)^2}$$

e.g. Find $\frac{d}{dx}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right)$

$$\frac{d}{dx}\left(\frac{1}{\sqrt{x}}+\sqrt{x}\right) = \frac{d}{dx}\left(x^{-\frac{1}{2}}+x^{\frac{1}{2}}\right)$$

$$= -\frac{1}{2}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}$$

Derivatives of Trigonometric Function :

Preparations :

$$\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2(\frac{x}{2})}{x^2}$$

Note: $\cos x = 1-2\sin^2(\frac{x}{2})$

$\therefore\ 1-\cos x = 2\sin^2(\frac{x}{2})$

$$= \lim_{x\to 0}\frac{1}{2}\frac{\sin^2(\frac{x}{2})}{(\frac{x}{2})^2}$$

$$= \frac{1}{2}$$

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot x$$

$$= \lim_{x\to 0}\frac{1-\cos x}{x^2}\cdot\lim_{x\to 0}x$$

$$= \frac{1}{2}\cdot 0$$

$$= 0$$

Let $f(x) = \cos x$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\cos(x+\Delta x)-\cos x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{\cos x\cos\Delta x - \sin x\sin\Delta x - \cos x}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\cos x\frac{\cos\Delta x-1}{\Delta x} - \sin x\frac{\sin\Delta x}{\Delta x}$$

$$= -\sin x$$

$\left(\because \lim_{\Delta x\to 0}\frac{\cos\Delta x-1}{\Delta x} = 0 \text{ and } \lim_{\Delta x\to 0}\frac{\sin\Delta x}{\Delta x} = 1.\right)$

$$\therefore \frac{d}{dx}\cos x = -\sin x$$

Ex: Show $\frac{d}{dx}\sin x = \cos x$ by using method similar to the above.

$$\tan x = \frac{\sin x}{\cos x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x}$$

$$\vdots \qquad \text{Ex: By quotient rule.}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

Ex: $\frac{d}{dx}\sec x = \sec x \tan x$

$\frac{d}{dx}\csc x = -\csc x \cot x$

$\frac{d}{dx}\cot x = -\csc^2 x$

Derivative of e^x:

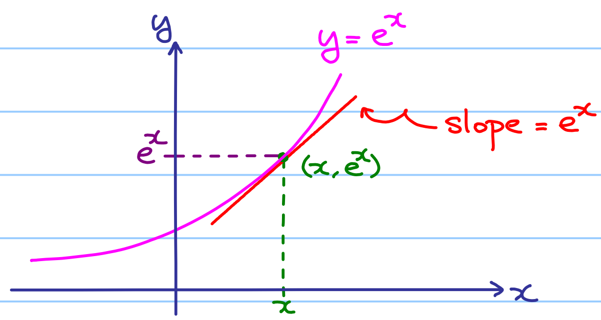
Recall: $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

Cheating: $\frac{d}{dx}e^x = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)$

$= 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$= e^x$ (getting back itself)

Geometrical meaning:



e.g. Find $\frac{d}{dx}[e^x(3x^2+7x-2)]$

$$\frac{d}{dx}[e^x(3x^2+7x-2)] = \left[\frac{d}{dx}e^x\right](3x^2+7x-2) + e^x\left[\frac{d}{dx}(3x^2+7x-2)\right]$$

$$= e^x(3x^2+7x-2) + e^x(6x+7)$$

$$= e^x(3x^2+13x+5)$$

Question: How to differentiate a more complicated function, such as $\sqrt{x^2+3x}$?

We need a tool called chain rule.

Chain Rule :

If $f(x)$ and $g(x)$ are differentiable function, then the composite function $(f\circ g)(x) = f(g(x))$ is also differentiable and

$$(f\circ g)'(x) = f'(g(x))\, g'(x) .$$

Hard to understand ? Let's rewrite :

Let $u = g(x)$, $y = f(u) = f(g(x))$, then

Chain rule : $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

Think as : $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$

e.g. Find the derivative of $\sqrt{x^2+3x}$.

Let $u = g(x) = x^2+3x$, $\quad \dfrac{du}{dx} = 2x+3$

$y = f(u) = \sqrt{u}$ $\quad \dfrac{dy}{du} = \dfrac{1}{2\sqrt{u}}$

then $f(g(x)) = \sqrt{x^2+3x}$

By chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$= \dfrac{1}{2\sqrt{u}} \cdot (2x+3)$

$= \underbrace{\dfrac{1}{2\sqrt{x^2+3x}}}_{f'(g(x))} \cdot \underbrace{(2x+3)}_{g'(x)}$ put $u = x^2+3x$ back

differentiate f
then put back $g(x)$

e.g. Find the derivative of $(3x^2-2x)^{2015}$.

Let $u = g(x) = 3x^2-2x$ $\quad \dfrac{du}{dx} = 6x-2$

$y = f(u) = u^{2015}$ $\quad \dfrac{dy}{du} = 2015u^{2014}$

then $f(g(x)) = (3x^2-2x)^{2015}$

By chain rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

$= 2015u^{2014} \cdot (6x-2)$

$= 2015(3x^2-2x)^{2014} \cdot (6x-2)$ put $u = 3x^2-2x$ back

$= 4030(3x^2-2x)^{2014} \cdot (3x-1)$

Slogan : differentiate layer by layer.

Ex : By using chain rule, show that $\frac{d}{dx} e^{ax} = ae^{ax}$

Ex : Find the derivative of $\left(\frac{x}{x+1}\right)^2$.

(a) By chain rule ;

(b) Write $\left(\frac{x}{x+1}\right)^2 = \frac{x^2}{(x+1)^2}$, then by quotient rule.

Ans : Both equal to $\frac{2x}{(x+1)^3}$.

e.g. Find the derivative of $e^{\sqrt{x^2+1}}$.

1st layer $y = e^w$ $\quad w = \sqrt{x^2+1}$

2nd layer $w = \sqrt{u}$ $\quad u = x^2+1$

3rd layer $u = x^2+1$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$= e^{\sqrt{x^2+1}} \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x$$

$$= \frac{xe^{\sqrt{x^2+1}}}{\sqrt{x^2+1}}$$

e.g. Revisit of quotient rule.

$$\left(\frac{f}{g}\right)'(x) = \frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}\left(f(x)[g(x)]^{-1}\right)$$

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\frac{d}{dx}[g(x)]^{-1} \qquad \text{(Product rule)}$$

Apply chain rule here

$$= \frac{df}{dx}[g(x)]^{-1} + f(x)\left\{-[g(x)]^{-2}\frac{dg}{dx}\right\}$$

$$= \frac{\frac{df}{dx}g(x) - f(x)\frac{dg}{dx}}{[g(x)]^2}$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Differentiability :

Theorem : If $f(x)$ is differentiable at $x = x_0$, then $f(x)$ is continuous at $x = x_0$.

proof : By assumption , $\displaystyle\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$ exists.

Also, we know $\displaystyle\lim_{\Delta x \to 0} \Delta x = 0$

$$\lim_{\Delta x \to 0} f(x_0+\Delta x) - f(x_0) = \lim_{\Delta x \to 0} \left(\frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \cdot \Delta x \right)$$

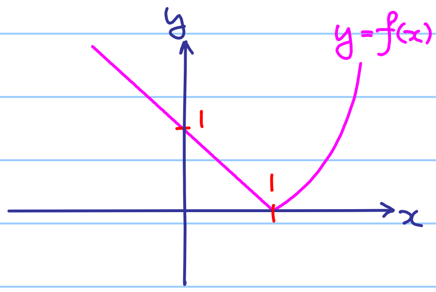
both exist

$$= \left(\lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \right) \cdot \left(\lim_{\Delta x \to 0} \Delta x \right)$$

$$= f'(x_0) \cdot 0 = 0$$

$\therefore \displaystyle\lim_{\Delta x \to 0} f(x_0+\Delta x) = f(x_0)$, so $f(x)$ is continuous at $x = x_0$.

However, the converse is NOT true.



e.g. If $f(x) = \begin{cases} x^2 - 1 & \text{if } x \geq 1 \\ 1 - x & \text{if } x < 1 \end{cases}$

$$\lim_{\Delta x \to 0^+} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{[(1+\Delta x)^2 - 1] - [1^2 - 1]}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{2\Delta x + \Delta x^2}{\Delta x} = 2$$

(it means we are looking at small but positive Δx)

$$\lim_{\Delta x \to 0^-} \frac{f(1+\Delta x) - f(1)}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{[1 - (1+\Delta x)] - [1^2 - 1]}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

(it means we are looking at small but negative Δx)

$$\lim_{\Delta x \to 0^+} \frac{f(1+\Delta x) - f(1)}{\Delta x} \neq \lim_{\Delta x \to 0^-} \frac{f(1+\Delta x) - f(1)}{\Delta x} \Rightarrow \lim_{\Delta x \to 0} \frac{f(1+\Delta x) - f(1)}{\Delta x} \text{ does NOT exist !}$$

$\therefore f(x)$ is NOT differentiable at $x = 1$.

Ex :

a) Show that $f(x)$ is continuous at $x=1$, i.e. $\lim_{x\to 1} f(x) = f(1)$.

(Therefore, $f(x)$ is continuous at $x=1$, but NOT differentiable at $x=1$)

b) Show that $f(x)$ is differentiable everywhere except $x=1$, and

$$f'(x) = \begin{cases} 2x & \text{if } x>1 \\ \text{undefined} & \text{if } x=1 \\ -1 & \text{if } x<1 \end{cases}$$

e.g. Let $f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Does $f'(0)$ exist ?

$$\lim_{\Delta x \to 0} \frac{f(0+\Delta x) - f(0)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x^2 \cos \frac{1}{\Delta x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \Delta x \cos \frac{1}{\Delta x}$$

$$= 0$$

By sandwich theorem

If $x \neq 0$, $f'(x) = 2x\cos\frac{1}{x} + x^2(-\sin\frac{1}{x})(-\frac{1}{x^2})$

$= 2x\cos\frac{1}{x} + \sin\frac{1}{x}$

$\therefore f$ is a differentiable function, i.e. differentiable at every point.

Note : It is wrong to say $f'(x) = 2x\cos\frac{1}{x} + \sin\frac{1}{x}$, so $f'(0)$ does NOT exist .

Now, $f'(x) = \begin{cases} 2x\cos\frac{1}{x} + \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Ex : Show $\lim_{x \to 0} f'(x)$ does NOT exist ($\Rightarrow f'(x)$ is NOT continuous at $x = 0$)

$\therefore f$ is differentiable ("good" in some sense)

but $f'(x)$ can be "bad"

e.g. Let $f: \mathbb{R} \to \mathbb{R}$ be a non-constant function such that

(i) f is differentiable at some $x_0 \in \mathbb{R}$

(ii) $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Show that :

a) $f(x) \neq 0$ for all $x \in \mathbb{R}$ and $f(0) = 1$.

b) f is differentiable at every $x \in \mathbb{R}$ and $f'(x) = \frac{f'(x_0)}{f(x_0)} f(x)$.

a) If $f(a)=0$ for some $a \in \mathbb{R}$

then for any $x \in \mathbb{R}$, we have

$f(x) = f(x-a+a) = f(x-a)f(a) = 0$

i.e. $f(x)$ is constant zero (Contradict to the assumption)

$\therefore f(x) \neq 0 \quad \forall x \in \mathbb{R}$.

Putting $x=y=0$,

$$f(0+0) = f(0)f(0)$$

$$f(0) = [f(0)]^2$$

$$f(0) = 1 \quad \text{or} \quad 0 \text{ (rejected)}$$

b) f is differentiable at x_0

$$\Rightarrow f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0)f(\Delta x) - f(x_0)}{\Delta x}$$

$$\therefore \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(x_0)}{\Delta x} = \frac{f'(x_0)}{f(x_0)} \qquad (\because f(x_0) \neq 0)$$

Now, $\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$= \lim_{\Delta x \to 0} \frac{f(x)f(\Delta x) - f(x)}{\Delta x}$$

$$= f(x) \cdot \lim_{\Delta x \to 0} \frac{f(\Delta x) - f(x_0)}{\Delta x}$$

$$= \frac{f'(x_0)}{f(x_0)} f(x)$$

$\therefore f$ is differentiable at every $x \in \mathbb{R}$ and $f'(x) = \frac{f'(x_0)}{f(x_0)} f(x)$.

(In fact, $f(x) = e^{ax}$ for some non-zero constant a.)

Ex: Let f be a differentiable function such that

$$f(x+y) = f(x) + f(y) + 3xy(x+y) \quad \forall x, y \in \mathbb{R}.$$

a) Show that $f'(0) = \lim_{\Delta x \to 0} \frac{f(\Delta x)}{\Delta x}$.

b) Hence, show that $f'(x) = f'(0) + 3x^2$.

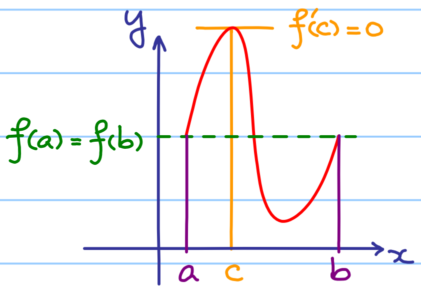
(In fact, $f(x) = f'(0)x + x^3$)

Rolle's Theorem

If f is a function such that :

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)
3) $f(a) = f(b)$

then there exists $c \in (a,b)$ such that $f'(c) = 0$.



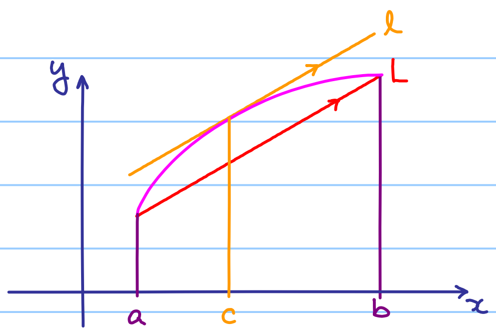
Mean Value Theorem

If f is a function such that :

1) f is continuous on $[a,b]$
2) f is differentiable on (a,b)

then there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b) - f(a)}{b-a}$

slope of ℓ slope of L.



proof : Let $g(x) = f(x)(b-a) - x[f(b) - f(a)]$

Check : 1) g is continuous on $[a,b]$

2) g is differentiable on (a,b)

3) $g(a) = g(b) = bf(a) - af(b)$

Apply Rolle's Theorem to g, the result follows.

e.g. A vehicle is speeding on a highway if its speed ≥ 120 km/h (at some moment)

If the length of the highway is 30 km and if Kelvin only spent 15 minutes on the highway. Should he be arrested?

Theorem :

If $f:\mathbb{R}\to\mathbb{R}$ is a differentiable and $f'(x)=0 \quad \forall x\in\mathbb{R}$,
then $f(x)$ is a constant function.

proof : Fix $x_0\in\mathbb{R}$, let $x\in\mathbb{R}\setminus\{x_0\}$

If $x>x_0$, note f is differentiable everywhere (in particular, on (x_0,x))

$\Rightarrow f$ is continuous everywhere (in particular, on $[x_0,x]$)

Apply MVT, $\exists\, c\in(x_0,x)$ such that

$$f(x_0)-f(x)=\underbrace{f'(c)}_{\substack{\parallel \\ 0 \text{ by assumption.}}}(x-x_0)=0$$

i.e. $f(x)=f(x_0) \quad \forall x>x_0$

We have similar result if $x<x_0$, the result follows.

e.g. Let $f(x)=\cos^2 x+\sin^2 x$

$f'(x)=-2\cos x\sin x+2\sin x\cos x=0$

$\therefore \cos^2 x+\sin^2 x$ is a constant.

In particular, $f(0)=1$, so $f(x)=\cos^2 x+\sin^2 x=1$

Theorem :

If $f,g:\mathbb{R}\to\mathbb{R}$ are differentiable and $f'(x)=g'(x) \quad \forall x\in\mathbb{R}$,
then $f(x)=g(x)+C$, where C is a constant.

proof : Let $h(x)=f(x)-g(x)$.

Then $h'(x)=f'(x)-g'(x)=0$

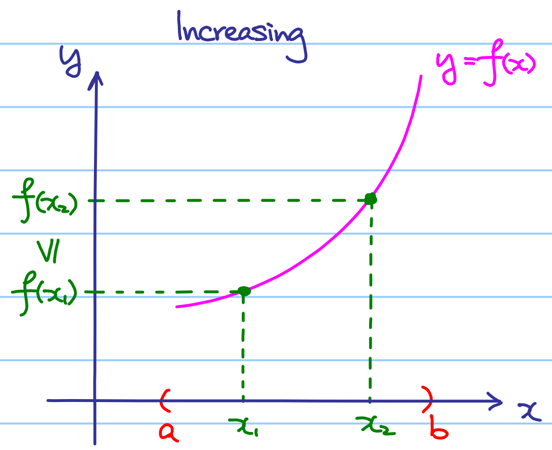
$\therefore h(x)=C$, where C is a constant. i.e. $f(x)=g(x)+C$.

Next, we are going to discuss how differentiation helps to find maximum / minimum points of a function.

Firstly, we make some preparations :

Increasing / Decreasing Functions

If $f(x)$ is a function such that for all x_1, x_2 with $a < x_1 < x_2 < b$, we have
† $f(x_1) \le f(x_2)$ ($f(x_1) \ge f(x_2)$), then $f(x)$ is called an increasing (a decreasing) function on (a,b).

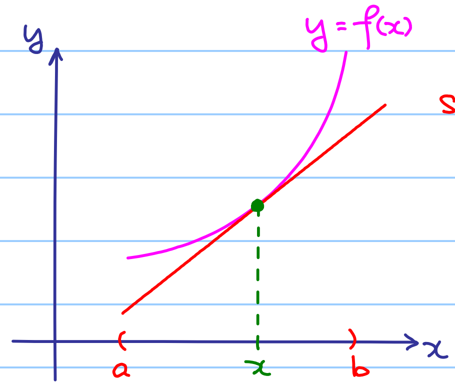


Roughly speaking:
The larger x we input
the larger y we get!

† If we have strict inequality, it is called a strictly increasing (decreasing) function on (a,b).

Theorem:

If $f(x)$ is differentiable on (a,b) and †† $f'(x) \ge 0$ ($f'(x) \le 0$) for all $x \in (a,b)$, then $f(x)$ is increasing (decreasing) on (a,b).



†† If we have strict inequality, $f(x)$ is a strictly increasing (decreasing) function on (a,b).

proof: If $a < x_1 < x_2 < b$,

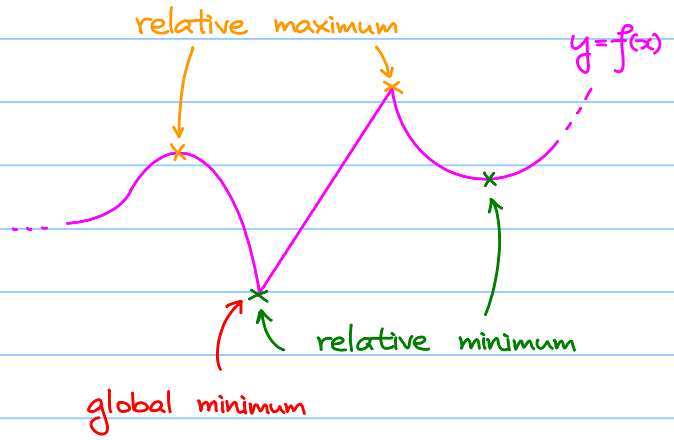
apply MVT to f on $[x_1, x_2]$,

$\exists\, c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = \underbrace{f'(c)}_{\geq 0} \underbrace{(x_2 - x_1)}_{> 0} \geq 0$

By assumption

Relative / Global Extrema:

Idea:



Note: No global maximum in this case.

f has a global maximum (resp. minimum) point at a if

$f(x) \leq f(a)$ (resp. $f(x) \geq f(a)$) for all x in the domain of f.

f has a relative maximum (resp. minimum) point at a if

$f(x) \leq f(a)$ (resp. $f(x) \geq f(a)$) for all x in a neighborhood of a.

e.g. Number of days of using drug : x

Life of a fish : T (weeks) which is estimated by

$T(x) = -5x^2 + 80x - 120$

$T'(x) = -10x + 80$

$T'(x) > 0$ $\qquad$ $T'(x) < 0$

$-10x + 80 > 0$ $\qquad$ $-10x + 80 < 0$

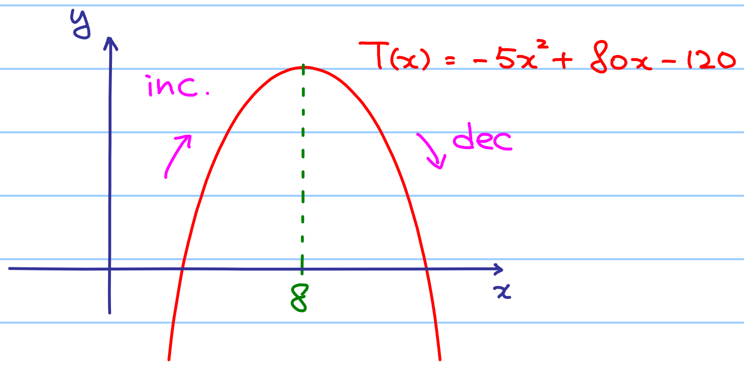
$x < 8$ $\qquad$ $x > 8$

$\therefore$ $T(x)$ is strictly increasing when $x < 8$ and

$T(x)$ is strictly decreasing when $x > 8$.

Not hard to understand why $T(x)$ attains maximum when $x = 8$

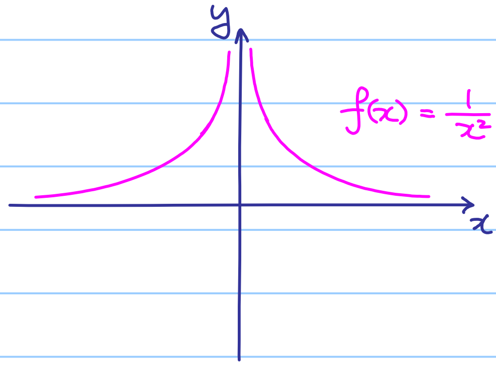
and maximum life of a fish $= T(8) = 200$ (weeks)



Note : $T'(8) = 0$.

Remark: Verify the above result by completing square.

e.g. Let $f(x) = \frac{1}{x^2}$, $x \neq 0$



$f'(x) = \frac{-2}{x^3}$

$f'(x) > 0$ if $x < 0$

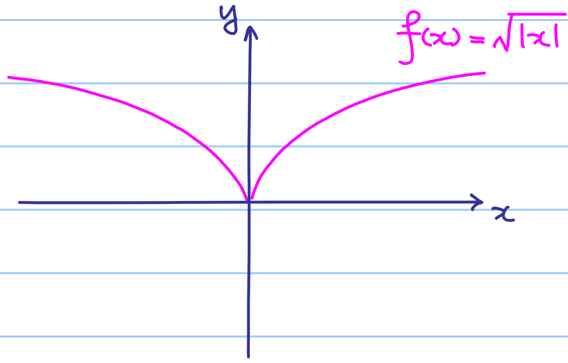
$f'(x) < 0$ if $x > 0$

$\therefore$ $f(x)$ is strictly increasing when $x<0$

$f(x)$ is strictly decreasing when $x>0$

However, $f(0)$ is NOT well-defined, so there is NO maximum point.

e.g. Let $f(x) = \sqrt{|x|}$



Rewrite :

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x>0 \\ 0 & \text{if } x=0 \\ \sqrt{-x} & \text{if } x<0 \end{cases}$$

If $x>0$, $f(x) = \sqrt{x}$, then $f'(x) = \frac{1}{2\sqrt{x}} > 0$

If $x<0$, $f(x) = \sqrt{-x}$, then $f'(x) = -\frac{1}{2\sqrt{-x}} < 0$

$\therefore$ $f(x)$ is strictly increasing when $x>0$

$f(x)$ is strictly decreasing when $x<0$

However, $\lim\limits_{\Delta x \to 0^+} \frac{f(0+\Delta x)-f(0)}{\Delta x} = \lim\limits_{\Delta x \to 0^+} \frac{\sqrt{\Delta x}}{\Delta x} = \lim\limits_{\Delta x \to 0^+} \frac{1}{\sqrt{\Delta x}}$ which does NOT exist,

$\Rightarrow$ $\lim\limits_{\Delta x \to 0} \frac{f(0+\Delta x)-f(0)}{\Delta x}$ does NOT exist

$\Rightarrow$ $f'(0)$ does NOT exist

but as we can see f still attains minimum at $x=0$.

$\therefore$ Solving $f'(x) = 0$ to find max / min is NOT enough.

Exact statement :

1st Derivative Check :

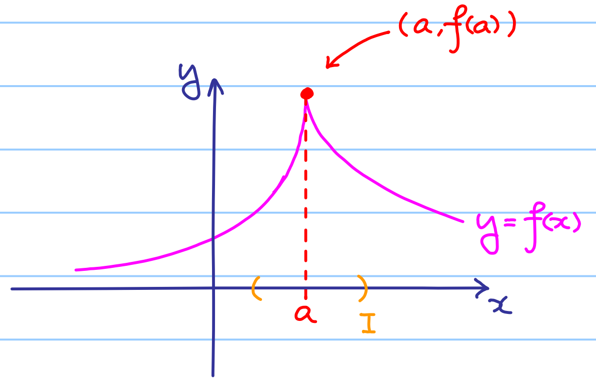
Suppose $f(x)$ is continuous at $x=a$ and differentiable on some neighborhood I containing a, except possibly at $x=a$ itself.

If $f'(x) \geq 0$ for all x in I with $x<a$, and

$f'(x) \leq 0$ for all x in I with $x>a$,

then $(a, f(a))$ is a relative maximum.

(Similar for relative minimum)



(Remember the slogan : Change sign of $f'(x)$ at $x=a$)

proof : If $x \in I$ and $x<a$,

apply MVT to f on $[x,a]$,

$\exists\, c \in (x,a)$ such that $f(a)-f(x) = \underbrace{f'(c)}_{\geq 0}\underbrace{(a-x)}_{> 0} \geq 0$

By assumption

$\therefore f(x) \leq f(a) \quad \forall x \in I$ with $x<a$

Similarly, we have $f(x) \leq f(a) \quad \forall x \in I$ with $x>a$

$\therefore f(x) \leq f(a) \quad \forall x \in I$.