Recall :

X : Discrete Random Variable

$p(x)$: Probability distribution

Expected Value of X = $E(X) = \sum_{x:p(x)\neq 0} x\,p(x)$

Variance of X = $Var(X) = \sum_{x:p(x)\neq 0} (x-\mu)^2 p(x)$

Another Interpretation of Var(X)

probability of getting x

$$E(X) = \sum_{x:p(x)\neq 0} x\,p(x)$$

value assigned $= x$

probability of getting x

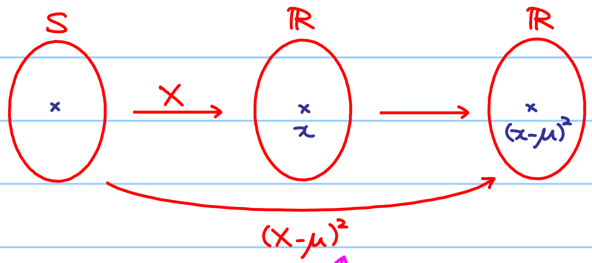
$$Var(X) = \sum_{x:p(x)\neq 0} (x-\mu)^2 p(x)$$

value assigned $= (x-\mu)^2$

probability of getting x

$$\mathrm{Var}(X) = \sum_{x:\,p(x)\neq 0} (x-\mu)^2 \, p(x)$$

value assigned $= (x-\mu)^2$



regarded as another random variable

$$\therefore \quad \mathrm{Var}(X) = E((X-\mu)^2)$$

Alternative Formula for Var(X)

$$\mathrm{Var}(X) = E((X-\mu)^2)$$

$$= \sum_{x : p(x) \neq 0} (x-\mu)^2 p(x)$$

$$
\begin{aligned}
Var(X) &= E((X-\mu)^2) \\
&= \sum_{x:p(x)\neq 0} (x-\mu)^2 p(x) \\
&= \sum_{x:p(x)\neq 0} (x^2-2x\mu+\mu^2) p(x) \\
&= \sum_{x:p(x)\neq 0} x^2 p(x) - \sum_{x:p(x)\neq 0} 2x\mu p(x) + \sum_{x:p(x)\neq 0} \mu^2 p(x) \\
&= \sum_{x:p(x)\neq 0} x^2 p(x) - 2\mu \sum_{x:p(x)\neq 0} x p(x) + \mu^2 \sum_{x:p(x)\neq 0} p(x) \\
&= E(X^2) - 2\mu\cdot\mu + \mu^2\cdot 1 \\
&= E(X^2) - \mu^2 \\
&= E(X^2) - E(X)^2
\end{aligned}
$$

e.g. Roll a die

Let X be the random variable denotes the number facing up.

$E(X) = \frac{7}{2} = 3.5$

$$Var(X) = (1-3.5)^2 \cdot \frac{1}{6} + (2-3.5)^2 \cdot \frac{1}{6} + (3-3.5)^2 \cdot \frac{1}{6} + (4-3.5)^2 \cdot \frac{1}{6} + (5-3.5)^2 \cdot \frac{1}{6} + (6-3.5)^2 \cdot \frac{1}{6}$$

$$= \frac{35}{12}$$

(Verification of the alternative formula)

$$E(X^2) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + 3^2 \cdot \frac{1}{6} + 4^2 \cdot \frac{1}{6} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{6}$$

$$= \frac{91}{6}$$

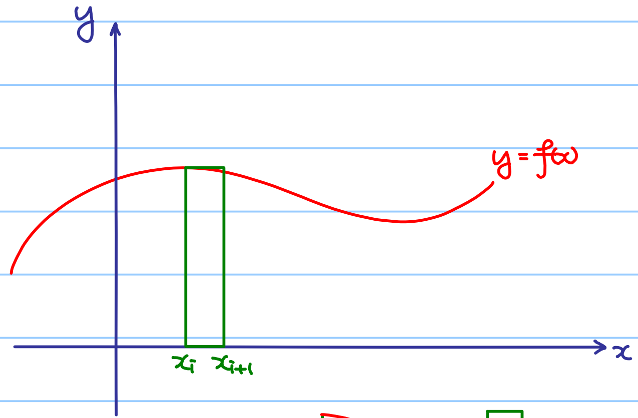
$$E(X)^2 = \left(\frac{7}{2}\right)^2 = \frac{49}{4}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

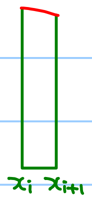
Expected Value and Variance of Continuous Random Variables

X : Continuous Random Variable

$f(x)$: Probability distribution



$P(x_i \le X \le x_{i+1})$ =



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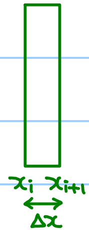

= $f(x_i)\,\Delta x$

$$E(X) = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{x_i}_{\text{value}} \cdot \underbrace{f(x_i)\,\Delta x}_{\text{probability}} = \int_{-\infty}^{+\infty} x f(x)\,dx$$

$$\therefore \quad \mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\,dx$$

Similarly,

$$Var(X) = \lim_{n\to\infty} \sum_{i=1}^{n} \underbrace{(x_i-\mu)^2}_{\text{value}} \cdot \underbrace{f(x_i)\,\Delta x}_{\text{probability}} = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx$$

$$\therefore \quad Var(X) = E((X-\mu)^2) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx$$

Alternative Formula for Var(X) :

$$Var(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} (x^2 - 2x\mu + \mu^2)\, f(x)\,dx$$

$$\mathrm{Var}(X) = \int_{-\infty}^{+\infty} (x-\mu)^2 f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} (x^2 - 2x\mu + \mu^2) f(x)\,dx$$

$$= \int_{-\infty}^{+\infty} x^2 f(x)\,dx - 2\mu \int_{-\infty}^{+\infty} x f(x)\,dx + \mu^2 \int_{-\infty}^{+\infty} f(x)\,dx$$

$$= E(X^2) - 2\mu \cdot \mu + \mu^2 \cdot 1$$

$$= E(X^2) - \mu^2$$

$$= E(X^2) - E(X)^2$$

Uniform Density Function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

and let X be random variable whose probability distribution is $f(x)$.

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$= \int_a^b \frac{x}{b-a}\, dx$$

$$= \left[\frac{x^2}{2(b-a)}\right]_a^b$$

$$= \frac{a+b}{2}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\, dx$$

$$= \int_a^b \frac{x^2}{b-a}\, dx$$

$$= \left[\frac{x^3}{3(b-a)}\right]_a^b$$

$$= \frac{b^2+ab+a^2}{3}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

Exponential Density Function

Let $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$, where $\lambda > 0$

and let X be random variable whose probability distribution is $f(x)$.

$$E(X) = \int_{-\infty}^{+\infty} x f(x)\, dx$$

$$= \int_{0}^{+\infty} \lambda x e^{-\lambda x}\, dx$$

why ?

$$= \int_{0}^{+\infty} x \, d(-e^{-\lambda x})$$

$$= \underbrace{\left[-x e^{-\lambda x}\right]_{0}^{+\infty}}_{=0} - \int_{0}^{+\infty} -e^{-\lambda x}\, dx$$

$$= \int_{0}^{+\infty} e^{-\lambda x}\, dx$$

$$= \left[-\frac{1}{\lambda} e^{-\lambda x}\right]_{0}^{+\infty}$$

$$= \frac{1}{\lambda}$$

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)\,dx = \int_0^{+\infty} \lambda x^2 e^{-\lambda x}\,dx \overset{Ex:}{=} \cdots = \frac{2}{\lambda^2}$$

$$E(X)^2 = \frac{1}{\lambda^2}$$

$$Var(X) = E(X^2) - E(X)^2 = \frac{1}{\lambda^2}$$