

Review :

1) Notations :

Set: collection of objects (elements)

\subseteq : subset

\in : belongs to

e.g. $S = \{1, 2, 3\}$

That means S is a set containing 3 elements, namely 1, 2 and 3.

OR : $1, 2, 3 \in S$

If $T = \{1, 2, 3, 4\}$, then we say S is a subset of T , or $S \subseteq T$.

That means all elements in S and also in T .

Notations often used in this course :

\mathbb{R} : set of all real numbers

$[a, b]$: set of all real numbers x such that $a \leq x \leq b$

(a, b) : set of all real numbers x such that $a < x < b$

$[a, +\infty)$: set of all real numbers x such that $a \leq x$

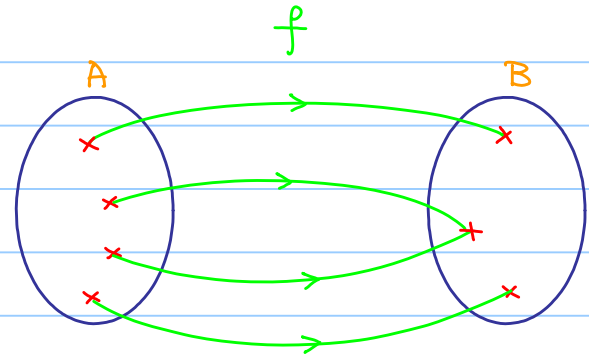
$\mathbb{R} \setminus \{a\}$: set of all real numbers except the number a

2) Functions :

Function : A function is a rule that assigns to each object in a set A exactly one object in a set B .

set A : domain (input)

set B : range (output)



A function f from A to B

We denote it by $f: A \rightarrow B$

e.g. Find the domain of the functions :

a) $f(x) = \frac{1}{x-3}$

b) $g(t) = \frac{\sqrt{3-2t}}{t^2+4}$

e.g. Find the domain of the functions :

a) $f(x) = \frac{1}{x-3}$

Ans: a) x can be any real number except 3

i.e. domain = $\mathbb{R} \setminus \{3\}$

b) $g(t) = \frac{\sqrt{3-2t}}{t^2+4}$

b) $3-2t \geq 0 \Rightarrow t \leq \frac{3}{2}$

i.e. domain = $(-\infty, \frac{3}{2}]$

e.g. What is the difference between $f(x) = \frac{x^2-1}{x-1}$ and $g(x) = x+1$?

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Ans : domain of $f = \mathbb{R} \setminus \{1\}$

domain of $g = \mathbb{R}$

e.g. Piecewise-defined function

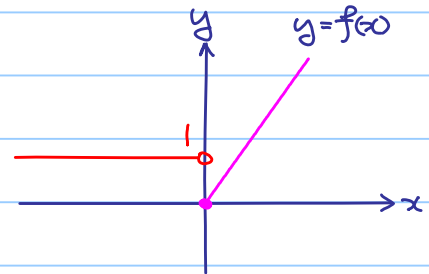
$f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

$$f(-1) = 1$$

$$f(0) = 0$$

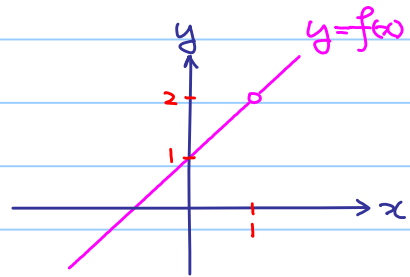
$$f(1) = 2$$



e.g. Let $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

We can rewrite f as the following:

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



eg. Composition of functions

$f, g: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = x^2 + 3x + 1, \quad g(x) = x + 1.$$

Think: $f(\Delta) = \Delta^2 + 3\Delta + 1$ Δ : input

$$\begin{aligned} \text{Then } f(g(x)) &= (x+1)^2 + 3(x+1) + 1 && (\text{Now: input} = \Delta = g(x) = x+1) \\ &= x^2 + 5x + 5 \end{aligned}$$

What is $g(f(x))$? Ans: $g(f(x)) = x^2 + 3x + 2$

Sometimes, we write $(f \circ g)(x)$ instead of $f(g(x))$ to emphasize it depends on x .

Functional Models :

Real World Problem :

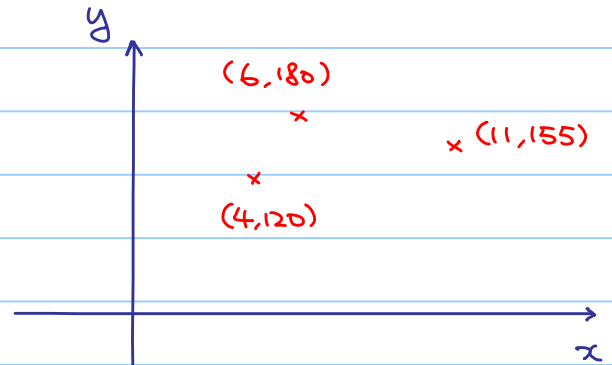
x : number of products produced (in thousands)

$P(x)$: profit (in thousands of dollars) (assumption : depending on x only)

Aim : Maximize the profit !

Step 1 : Observation

x	$P(x)$
4	120
6	180
11	155

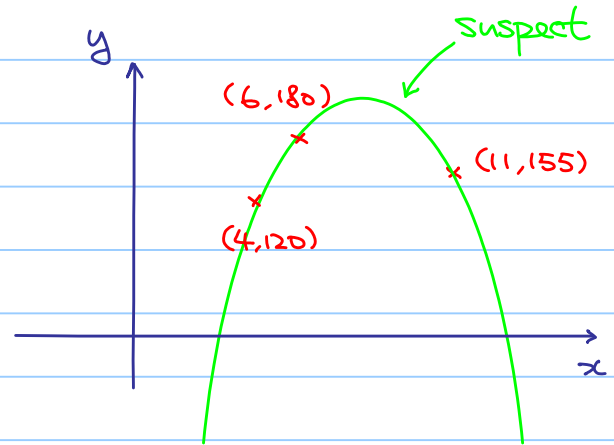


Step 2: Modelling

Suspect the formula behind to be $P(x) = ax^2 + bx + c$

Ex: Use the data above
to solve a , b and c .

Ans: $a = -5$, $b = 80$, $c = -120$



Step 3 : Prediction

Maximizing profit : $P(x) = -5x^2 + 80x - 120$
 $= -5(x-8)^2 + 200$

\therefore Maximum profit = 200 can be attained when $x=8$

Break-even : $P(x) = 0$
 $-5x^2 + 80x - 120 = 0$
 $x = 8 \pm 2\sqrt{10}$

Step 4 : Testing the model

Accept, Modify or Reject ?

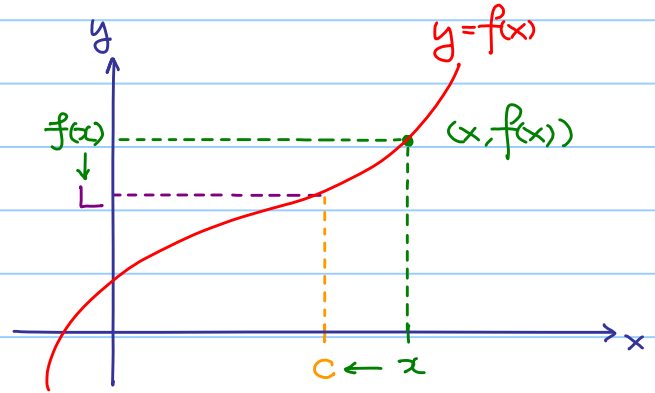
Question : How to find max / min for a general function ?

Answer : Calculus helps (also other applications)

Limits of Functions :

Limit of a function :

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer[†] to c from both sides, then L is called the limit of $f(x)$ at c . We write $\lim_{x \rightarrow c} f(x) = L$

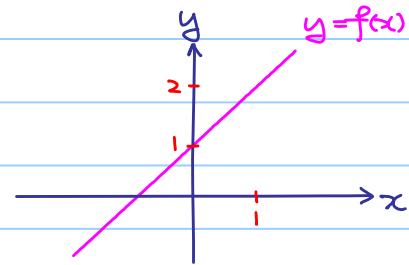


† Note : a little bit misleading !

$f(c)$ may NOT equal to L , even it may be undefined !

e.g. If $f(x) = x+1$, find $\lim_{x \rightarrow 1} f(x)$

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	2	2.001	2.01	2.1



$f(x)$ tends to 2 as x tends to 1.

We write $\lim_{x \rightarrow 1} f(x) = 2$.

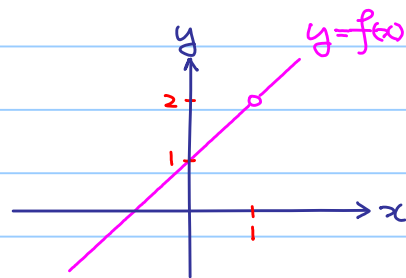
Remarks:

- 1) \dagger The table only gives an intuitive idea, but **NOT** a rigorous proof!
- 2) Do **NOT** regard as putting $x=1$ into $f(x)$ and get $f(1)=2$!

e.g. Let $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

We can rewrite f as the following:

$$f(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ \text{undefined} & \text{if } x = 1 \end{cases}$$



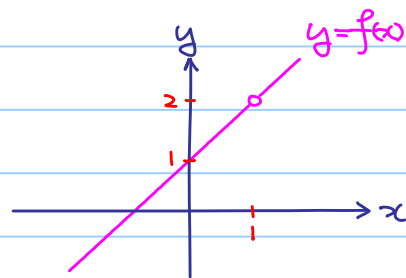
graph of f

x	0.9	0.99	0.999		1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	undefined	2.001	2.01	2.1

e.g. Let $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x^2-1}{x-1}$, $x \neq 1$.

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graph of f

$f(x)$ tends to 2 as x tends to 1

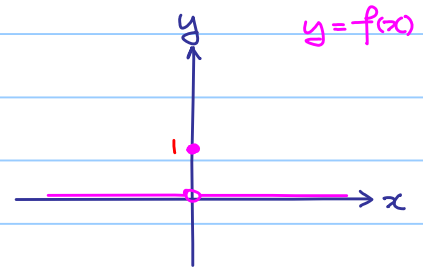
(But, we do NOT care what happens when $x=1$!)

We write $\lim_{x \rightarrow 1} f(x) = 2$.

Compare with the previous example !

e.g. If $f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$,

find $\lim_{x \rightarrow 0} f(x)$

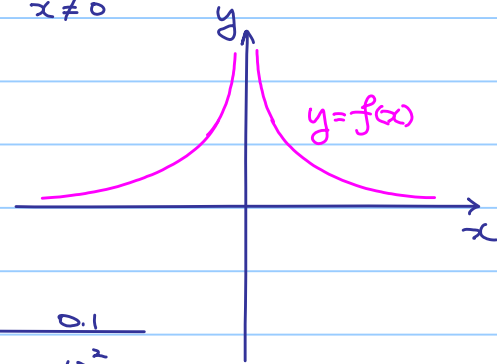


x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	0	0	0	1	0	0	0

Do NOT care!

$\lim_{x \rightarrow 0} f(x) = 0$ which does NOT equal to $f(0) = 1$.

eg. Let $f: \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$, $x \neq 0$



x	-0.1	-0.01	-0.001	0	0.001	0.01	0.1
$f(x)$	10^2	10^4	10^6	undefined	10^6	10^4	10^2

$f(x)$ tends to $+\infty$ (NOT a real number) as x tends to 0

$\therefore \lim_{x \rightarrow 0} f(x)$ does NOT exist.

(But, some still write $\lim_{x \rightarrow 0} f(x) = +\infty$.)

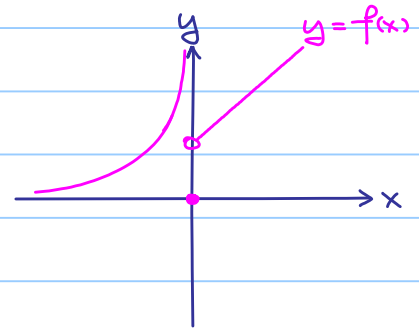
Right Hand Limit and Left Hand Limit :

If $f(x)$ gets closer and closer to a real number L as x gets closer and closer to c from right (resp. left) hand side, then L is called the right (resp. left) hand limit of $f(x)$ at c .

We write $\lim_{x \rightarrow c^+} f(x) = L$ (resp. $\lim_{x \rightarrow c^-} f(x) = L$)

e.g.

$$f(x) = \begin{cases} x+1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ \frac{1}{x^2} & \text{if } x < 0 \end{cases}$$



$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x+1 = 1$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{1}{x^2} \quad (\text{does NOT exist})$$

$$f(0) = 0$$

Remark:

Right hand limit and left hand limit of a function at a point is **NOT** necessary to be the same!

FACT :

$\lim_{x \rightarrow c} f(x) = L$ if and only if $\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^-} f(x) = L$
(i.e. both right and left hand limit exist and equal to L .)

FACT (without proof)

(1) If k is a constant, $\lim_{x \rightarrow c} k = k$

(2) $\lim_{x \rightarrow c} x = c$

↑
regarded as constant function $f(x) = k$

Algebraic Properties of Limits :

If $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist \leftarrow (very important !), then

$$(1) \lim_{x \rightarrow c} (f(x) + g(x)) = \lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)$$

$$(2) \lim_{x \rightarrow c} (f(x) - g(x)) = \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} g(x)$$

$$(3) \lim_{x \rightarrow c} (f(x)g(x)) = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x)$$

$$(4) \lim_{x \rightarrow c} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)} \quad \text{if } \lim_{x \rightarrow c} g(x) \neq 0$$

e.g. Find $\lim_{x \rightarrow 2} 3x^2 - 5$

e.g. Find $\lim_{x \rightarrow 2} 3x^2 - 5$

① $\lim_{x \rightarrow 2} x = 2$, so $\lim_{x \rightarrow 2} x^2 = \lim_{x \rightarrow 2} (x \cdot x) = \lim_{x \rightarrow 2} x \cdot \lim_{x \rightarrow 2} x = 2 \cdot 2 = 4$

By (3)
↓

② $\lim_{x \rightarrow 2} 3 = 3$, $\lim_{x \rightarrow 2} x^2 = 4$, so $\lim_{x \rightarrow 2} 3x^2 = \lim_{x \rightarrow 2} 3 \cdot \lim_{x \rightarrow 2} x^2 = 3 \cdot 4 = 12$

By (3)
↓

③ $\lim_{x \rightarrow 2} 3x^2 = 12$, $\lim_{x \rightarrow 2} 5 = 5$, so $\lim_{x \rightarrow 2} 3x^2 - 5 = \lim_{x \rightarrow 2} 3x^2 - \lim_{x \rightarrow 2} 5 = 12 - 5 = 7$

By (2)
↓

But what we write :

$$\lim_{x \rightarrow 2} 3x^2 - 5 = 3(\lim_{x \rightarrow 2} x)^2 - 5 = 7$$

e.g. Find $\lim_{x \rightarrow 1} \frac{3x^2 - 8}{x - 2}$

$$\lim_{x \rightarrow 1} \frac{3x^2 - 8}{x - 2} = \frac{3(\lim_{x \rightarrow 1} x)^2 - 8}{(\lim_{x \rightarrow 1} x) - 2} = \frac{3(1) - 8}{1 - 2} = 5$$

e.g. Think:

$$\lim_{x \rightarrow 0} \frac{1}{x} = \lim_{x \rightarrow 0} x \cdot \frac{1}{x^2} = \lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \frac{1}{x^2} = 0$$

\parallel
 0

\uparrow
 $(*)$

But we know $\lim_{x \rightarrow 0} \frac{1}{x}$ does NOT exist.

What's wrong?

e.g. Think:

$$\lim_{x \rightarrow 0} \frac{1}{x} = \lim_{x \rightarrow 0} x \cdot \frac{1}{x^2} = \lim_{x \rightarrow 0} x \cdot \lim_{x \rightarrow 0} \frac{1}{x^2} = 0$$

\parallel
 0

\uparrow
 $(*)$

But we know $\lim_{x \rightarrow 0} \frac{1}{x}$ does NOT exist.

What's wrong?

Ans: $\lim_{x \rightarrow 0} \frac{1}{x}$ does NOT exist, so we cannot use (3) at $(*)$.

e.g. Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 3x + 2}$

Note $\lim_{x \rightarrow 1} x^2 - 3x + 2 = 0$, so we cannot use (4).

$$\lim_{x \rightarrow 1} \frac{x^2 - 1}{x^2 - 3x + 2} = \lim_{x \rightarrow 1} \frac{(x-1)(x+1)}{(x-1)(x-2)} = \lim_{x \rightarrow 1} \frac{x+1}{x-2} \stackrel{\text{By (4)}}{=} \frac{\lim_{x \rightarrow 1} x+1}{\lim_{x \rightarrow 1} x-2} = \frac{2}{-1} = -2$$

e.g. Let $f: \mathbb{R} \setminus \{1\} \rightarrow \mathbb{R}$ defined by $f(x) = \frac{\sqrt{x}-1}{x-1}$, $x \neq 1$.

Find $\lim_{x \rightarrow 1} f(x)$.

Note : For $x \neq 1$ ($x-1 \neq 0$, denominator is nonzero.)

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{x-1} \cdot \frac{\sqrt{x}+1}{\sqrt{x}+1} = \frac{1}{\sqrt{x}+1}$$

$$\lim_{x \rightarrow 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x \rightarrow 1} \frac{1}{\sqrt{x}+1} \quad (\text{We only concern those } x \text{ near } 1 \text{ but NOT equal to } 1)$$

$$= \frac{1}{2} \quad (\text{Still the same, do NOT regard as putting } x=1)$$

Limit at Infinity:

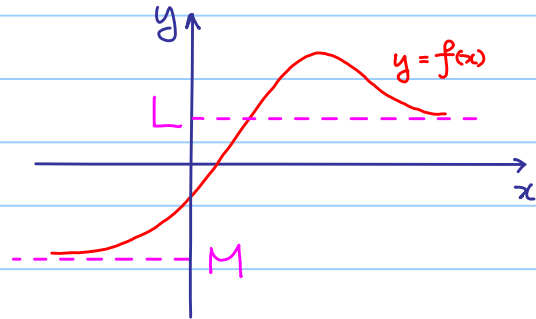
If $f(x)$ gets closer and closer to a real number L as x gets bigger and bigger (as x goes to $+\infty$), then L is called the limit of $f(x)$ at $+\infty$.

We write $\lim_{x \rightarrow +\infty} f(x) = L$.

(Similar definition for $\lim_{x \rightarrow -\infty} f(x)$)

$$\lim_{x \rightarrow +\infty} f(x) = L$$

$$\lim_{x \rightarrow -\infty} f(x) = M$$



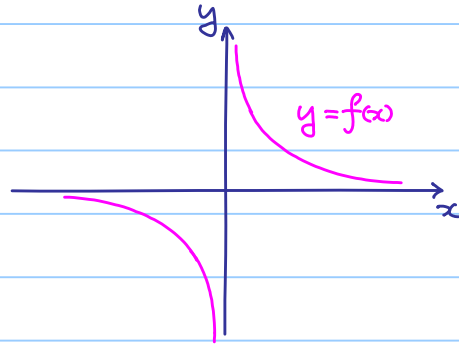
$\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$ are **NOT** necessary to be the same!

But if $\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = L$, some simply write $\lim_{x \rightarrow \infty} f(x) = L$.

e.g. $f(x) = \frac{1}{x}$

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow -\infty} f(x) = 0$$

OR simply $\lim_{x \rightarrow \infty} f(x) = 0$



FACT (without proof)

If $k > 0$, then $\lim_{x \rightarrow \infty} \frac{1}{x^k} = 0$

Algebraic Properties of Limits at Infinity :

If $\lim_{x \rightarrow +\infty} f(x)$ and $\lim_{x \rightarrow +\infty} g(x)$ exist \leftarrow (very important !), then

$$(1) \lim_{x \rightarrow +\infty} (f(x) + g(x)) = \lim_{x \rightarrow +\infty} f(x) + \lim_{x \rightarrow +\infty} g(x)$$

$$(2) \lim_{x \rightarrow +\infty} (f(x) - g(x)) = \lim_{x \rightarrow +\infty} f(x) - \lim_{x \rightarrow +\infty} g(x)$$

$$(3) \lim_{x \rightarrow +\infty} (f(x)g(x)) = \lim_{x \rightarrow +\infty} f(x) \cdot \lim_{x \rightarrow +\infty} g(x)$$

$$(4) \lim_{x \rightarrow +\infty} \left(\frac{f(x)}{g(x)} \right) = \frac{\lim_{x \rightarrow +\infty} f(x)}{\lim_{x \rightarrow +\infty} g(x)} \quad \text{if } \lim_{x \rightarrow +\infty} g(x) \neq 0$$

Similar results hold for limits at $-\infty$.

e.g. Find $\lim_{x \rightarrow \infty} \frac{3x^2}{x^2 + x + 1}$

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$$\begin{aligned} & \lim_{x \rightarrow \infty} \frac{3x^2}{x^2+x+1} \\ &= \lim_{x \rightarrow \infty} \frac{3}{1 + \frac{1}{x} + \frac{1}{x^2}} \\ &= \frac{3}{1+0+0} \\ &= 3 \end{aligned}$$

~~$\lim_{x \rightarrow \infty} \frac{3x^2}{x^2+x+1}$~~ \leftarrow Both limit do NOT exist!

e.g. Find $\lim_{x \rightarrow +\infty} \frac{2x+1}{3x^2-2x+1}$

e.g. Find $\lim_{x \rightarrow +\infty} \frac{2x+1}{3x^2-2x+1}$

$$\lim_{x \rightarrow +\infty} \frac{2x+1}{3x^2-2x+1}$$

$$= \lim_{x \rightarrow +\infty} \frac{\frac{2}{x} + \frac{1}{x^2}}{3 - \frac{2}{x} + \frac{1}{x^2}}$$

$$= \frac{0+0}{3-0+0}$$

$$= 0$$

Compare the previous two examples :

For the second example,

$2x+1$ and $3x^2-2x+1$ tend to $+\infty$ as x tends to $+\infty$.

But $3x^2-2x+1$ grows "faster" than $2x+1$.

↑
deg 2

↑
deg 1

Think: If $p(x)$ and $q(x)$ are polynomials

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_1 x + a_0 \quad \text{with } a_m > 0 \quad (\text{i.e. } \deg p(x) = m)$$

$$q(x) = b_n x^n + a_{n-1} x^{n-1} + \dots + b_1 x + b_0 \quad \text{with } b_n > 0 \quad (\text{i.e. } \deg q(x) = n)$$

then find $\lim_{x \rightarrow +\infty} \frac{p(x)}{q(x)}$ for the following cases :

1) $\deg p(x) > \deg q(x)$ i.e. $m > n$

2) $\deg p(x) = \deg q(x)$ i.e. $m = n$

3) $\deg p(x) < \deg q(x)$ i.e. $m < n$

Ans :

$$\lim_{x \rightarrow +\infty} \frac{p(x)}{q(x)} = \begin{cases} +\infty & \text{if } \deg p(x) > \deg q(x) \\ \frac{a_m}{b_m} & \text{if } \deg p(x) = \deg q(x) \\ 0 & \text{if } \deg p(x) < \deg q(x) \end{cases}$$