Solution to assignment 8

(1) (16.2, Q36):

From $(1,0)$ to $(0,1)$:

$$\mathbf{r}_1 = (1-t)\mathbf{i} + t\mathbf{j}, 0 \le t \le 1,$$
$$\frac{d\mathbf{r}_1}{dt} = -\mathbf{i} + \mathbf{j},$$
$$\mathbf{n}_1 |\mathbf{v}_1| = \mathbf{i} + \mathbf{j},$$
$$\mathbf{F} = (x+y)\mathbf{i} - \left(x^2+y^2\right)\mathbf{j} = \mathbf{i} - \left(1 - 2t + 2t^2\right)\mathbf{j},$$
$$\mathbf{F} \cdot \mathbf{n}_1 |\mathbf{v}_1| = 2t - 2t^2.$$

So we have

$$\text{Flux}_1 = \int_0^1 \left(2t - 2t^2\right) dt = \left[t^2 - \frac{2}{3}t^3\right]_0^1 = \frac{1}{3}.$$

From $(0,1)$ to $(-1,0)$:

$$\mathbf{r}_2 = -t\mathbf{i} + (1-t)\mathbf{j}, 0 \le t \le 1,$$
$$\frac{d\mathbf{r}_2}{dt} = -\mathbf{i} - \mathbf{j},$$
$$\mathbf{n}_2 |\mathbf{v}_2| = -\mathbf{i} + \mathbf{j},$$
$$\mathbf{F} = (x+y)\mathbf{i} - \left(x^2+y^2\right)\mathbf{j} = (1-2t)\mathbf{i} - \left(1 - 2t + 2t^2\right)\mathbf{j}$$
$$\mathbf{F} \cdot \mathbf{n}_2 |\mathbf{v}_2| = (2t-1) + \left(-1 + 2t - 2t^2\right) = -2 + 4t - 2t^2.$$

So we have

$$\text{Flux}_2 = \int_0^1 \left(-2 + 4t - 2t^2\right) dt = \left[-2t + 2t^2 - \frac{2}{3}t^3\right]_0^1 = -\frac{2}{3}.$$

From $(-1,0)$ to $(1,0)$:

$$\mathbf{r}_3 = (-1+2t)\mathbf{i}, 0 \le t \le 1,$$
$$\frac{d\mathbf{r}_3}{dt} = 2\mathbf{i},$$
$$\mathbf{n}_3 |\mathbf{v}_3| = -2\mathbf{j},$$
$$\mathbf{F} = (x+y)\mathbf{i} - \left(x^2+y^2\right)\mathbf{j} = (-1+2t)\mathbf{i} - \left(1 - 4t + 4t^2\right)\mathbf{j}$$
$$\mathbf{F} \cdot \mathbf{n}_3 |\mathbf{v}_3| = 2\left(1 - 4t + 4t^2\right).$$

So we have

$$\text{Flux}_3 = 2\int_0^1 \left(1 - 4t + 4t^2\right) dt = 2\left[t - 2t^2 + \frac{4}{3}t^3\right]_0^1 = \frac{2}{3}.$$

Thus the total flux is

$$\text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}.$$

(2) (16.2, Q43):
The slope of the line through (x, y) and the origin is $\frac{y}{x}$.
$\Rightarrow \mathbf{v} = x\mathbf{i} + y\mathbf{j}$ is a vector parallel to that line and pointing away from the origin.
$\Rightarrow \mathbf{F} = -\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ is the unit vector pointing toward the origin.

(3) (16.3, Q31):
(a) $\mathbf{F} = \nabla\left(x^3y^2\right) \Rightarrow \mathbf{F} = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$.
Let C_1 be the path from $(-1, 1)$ to $(0, 0)$, which is

$$\mathbf{r}_1 = (t-1)\mathbf{i} + (-t+1)\mathbf{j}, 0 \le t \le 1,$$

$$d\mathbf{r}_1 = dt\mathbf{i} - dt\mathbf{j},$$

$$\mathbf{F} = 3(t-1)^4\mathbf{i} - 2(t-1)^4\mathbf{j},$$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 = \int_0^1 \left[3(t-1)^4 + 2(t-1)^4\right] dt = 1.$$

Let C_2 be the path from $(0, 0)$ to $(1, 1)$, which is

$$\mathbf{r}_2 = t\mathbf{i} + t\mathbf{j}, 0 \le t \le 1,$$

$$d\mathbf{r}_2 = dt\mathbf{i} + dt\mathbf{j},$$

$$\mathbf{F} = 3t^4\mathbf{i} + 2t^4\mathbf{j},$$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = \int_0^1 \left(3t^4 + 2t^4\right) dt = \int_0^1 5t^4 dt = 1.$$

Thus we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F} \cdot d\mathbf{r}_2 = 2.$$

(b) Since $f(x, y) = x^3y^2$ is a potential function for $\mathbf{F}$,

$$\int_{(-1,1)}^{(1,1)} \mathbf{F} \cdot d\mathbf{r} = f(1, 1) - f(-1, 1) = 2.$$

(4) (16.3, Q32):
$(2x\cos y)_y = -2x\sin y = (-x^2\sin y)_x$
$\Rightarrow \mathbf{F}$ is conservative.
$\Rightarrow$ There exists an f so that $\mathbf{F} = \nabla f$.
$\frac{\partial f}{\partial x} = 2x\cos y$
$\Rightarrow f(x, y) = x^2\cos y + g(y)$

$\Rightarrow \frac{\partial f}{\partial y} = -x^2 \sin y + \frac{\partial g}{\partial y} = -x^2 \sin y$
$\Rightarrow \frac{\partial g}{\partial y} = 0$
$\Rightarrow f(x, y) = x^2 \cos y + C$
$\Rightarrow \mathbf{F} = \nabla\left(x^2 \cos y\right)$.
(a) $\int_C 2x \cos y dx - x^2 \sin y dy = \left[x^2 \cos y\right]_{(1,0)}^{(0,1)} = 0 - 1 = -1$.
(b) $\int_C 2x \cos y dx - x^2 \sin y dy = \left[x^2 \cos y\right]_{(-1,\pi)}^{(1,0)} = 1 - (-1) = 2$.
(c) $\int_C 2x \cos y dx - x^2 \sin y dy = \left[x^2 \cos y\right]_{(-1,0)}^{(1,0)} = 1 - 1 = 0$.
(d) $\int_C 2x \cos y dx - x^2 \sin y dy = \left[x^2 \cos y\right]_{(1,0)}^{(1,0)} = 1 - 1 = 0$.