MATH1010 University Mathematics
Mean value theorem

1. Prove that for any $x > 0$,
$$1 + \frac{x}{2} - \frac{x^2}{8} < \sqrt{1+x} < 1 + \frac{x}{2}.$$

2. Prove that for any $x \in (0, \frac{\pi}{2})$,
$$2 \ln \sec x \leq \sin x \tan x.$$

3. Prove that for any $x > 0$,
$$x - \frac{x^3}{3} < \tan^{-1} x.$$

4. Prove that for any $\theta > 0$,
$$1 - \frac{\theta^2}{2} < \cos\theta < 1 - \frac{\theta^2}{2} + \frac{\theta^4}{24}.$$

5. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that there exists $0 < x < 1$ such that
$$a_1 x + a_2 x^2 + \cdots + a_n x^n = \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}$$

6. Let $f(x)$ and $g(x)$ be functions which are continuous on $[a, b]$ and differentiable in (a, b). Suppose $f(a) = f(b) = 0$.
   (a) By considering the function $e^x f(x)$, show that there exists $\xi \in (a, b)$ such that $f'(\xi) + f(\xi) = 0$.
   (b) Prove that there exits $\eta \in (a, b)$ such that $f'(\eta) + g'(\eta) f(\eta) = 0$.

7. Let $f(x)$ be function, continuous on $[0, +\infty)$, differentiable on $(0, +\infty)$ which satisfies $f'(x) \leq f(x)$ for any $x > 0$. Prove that $f(x) \leq f(0)e^x$ for any $x > 0$.

8. Let $f(x)$ be a function which is twice differentiable on $\mathbb{R}$. Let a, b, c be real numbers with $a < b < c$. Let
$$F(x) = f(x) - \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) - \frac{(x-c)(x-a)}{(b-c)(b-a)} f(b) - \frac{(x-b)(x-a)}{(c-b)(c-a)} f(c).$$

(a) Prove that there exists $\zeta \in (a, b)$ such that $F'(\zeta) = 0$.

(b) Prove that there exists $\eta \in (a, c)$ such that

$$\frac{f''(\eta)}{2} = \frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-b)(c-a)}.$$

9. Suppose $0 < a < b$.

   (a) Prove that

   $$(1 + \ln a)(b - a) < b \ln b - a \ln a < (1 + \ln b)(b - a).$$

   (b) Prove that there exists $c \in (a, b)$ such that

   $$ae^b - be^a = (b - a)(c - 1)e^c.$$

10. Let $f(x)$ be a function such that $f''(x) < 0$ for any x.

    (a) Prove that $f'(x + 1) < f(x + 1) - f(x) < f'(x)$ for any x.

    (b) Prove that

    $$f'(1) + f'(2) + f'(3) < f(3) - f(0) < f'(0) + f'(1) + f'(2).$$

11. Let I be an open interval and $a, b \in I$ with $a < b$. If f is differentiable on I and if λ is a number between $f'(a)$ and $f'(b)$, show that there is at least one point $c \in (a, b)$ such that $f'(c) = \lambda$.

    (Hint: You may start with defining a function $f_a(t) = \begin{cases} f'(a) & \text{if } t = a \\ \frac{f(t)-f(a)}{t-a} & \text{if } t \neq a \end{cases}$.

12. Let $f(t)$ be a function on $\mathbb{R}$ with $f''(t) \geq 0$ for any $t \in \mathbb{R}$. Let $p, q > 1$ with $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

    (a) For any $x, y \in \mathbb{R}$ with $x < y$, let $z = \dfrac{x}{p} + \dfrac{y}{q}$.

       (i) Prove that $x < z < y$.

       (ii) Prove that there exists $x < \xi < z$ such that

       $$f'(\xi) = \frac{q(f(z) - f(x))}{y - x}.$$

(iii) Prove that
$$f(z) \le \frac{f(x)}{p} + \frac{f(y)}{q}.$$

(b) Let $a, b > 0$.

(i) Prove that
$$f(\ln(ab)) \le \frac{f(p\ln a)}{p} + \frac{f(q\ln b)}{q}.$$

(ii) Prove that
$$ab \le \frac{a^p}{p} + \frac{b^q}{q}.$$

13. Let $f(x)$ be a function which is differentiable on $(0, +\infty)$. Suppose

- $f'(x) > 0$ for any $x > 0$, and
- $\lim_{x\to 0^+} f(x) = 0$

(a) Prove that $f(x) > 0$ for any $x > 0$.

(b) Prove that $(1+x)\ln(1+x) - x\ln x > 0$ for any $x > 0$.

(c) Let $a > 0$ and $g(t) = \dfrac{\ln(1+a^t)}{t}$. Prove that $g'(t) < 0$ for any $t > 0$.

(d) Prove that $(u^q+v^q)^{\frac{1}{q}} < (u^p+v^p)^{\frac{1}{p}}$ for any $u, v, p, q > 0$ with $p < q$.

Solution:

1. Let $f(x) = 1 + \dfrac{x}{2} - \sqrt{1+x}$. Then $f(0) = 0$ and
$$f'(x) = \frac{1}{2} - \frac{1}{2\sqrt{1+x}} > 0$$
for $x > 0$. Thus $1 + \dfrac{x}{2} > \sqrt{1+x}$ for $x > 0$. On the other hand, let

$g(x) = \sqrt{1+x} - 1 - \dfrac{x}{2} + \dfrac{x^2}{8}$. Then $g(0) = 0$ and

$$\begin{aligned} g'(x) &= \frac{1}{2\sqrt{1+x}} - \frac{1}{2} + \frac{x}{4} \\ &> \frac{1}{2(1+\frac{x}{2})} - \frac{1}{2} + \frac{x}{4} \\ &= \frac{4 - 2(2+x) + x(2+x)}{4(2+x)} \\ &= \frac{x^2}{4(2+x)} \\ &> 0 \end{aligned}$$

for $x > 0$. Thus $1 + \dfrac{x}{2} - \dfrac{x^2}{8} < \sqrt{1+x}$ for $x > 0$.

2. Let $f(x) = \sin x \tan x - 2 \ln \sec x$. Then $f(0) = 0$ and for any $x \in (0, \frac{\pi}{2})$,

$$\begin{aligned} f'(x) &= \cos x \tan x + \sin x \sec^2 x - \frac{2 \sec x \tan x}{\sec x} \\ &= \sin x + \frac{\sin x}{\cos^2 x} - \frac{2 \sin x}{\cos x} \\ &= \frac{\sin x (\cos^2 x + 1 - 2\cos x)}{\cos^2 x} \\ &= \frac{\sin x (1 - \cos x)^2}{\cos^2 x} \\ &> 0 \end{aligned}$$

for $x \in (0, \frac{\pi}{2})$. Thus $\sin x \tan x > 2 \ln \sec x$ for $x \in (0, \frac{\pi}{2})$.

3. Let $f(x) = \tan^{-1} x - x + \dfrac{x^3}{3}$. Then $f(0) = 0$ and

$$f'(x) = \frac{1}{1+x^2} - 1 + x^2 = \frac{1 - (1+x^2) + x^2(1+x^2)}{1+x^2} = \frac{x^4}{1+x^2} > 0$$

for $x > 0$. Thus $f(x) > 0$ for $x > 0$. Therefore $\tan^{-1} x > x - \dfrac{x^3}{3}$ for $x > 0$.

4. Let $f(\theta) = \cos\theta - 1 + \dfrac{\theta^2}{2}$. Then

$$\begin{aligned} f'(\theta) &= \theta - \sin\theta \\ f''(\theta) &= 1 - \cos\theta \end{aligned}$$

Now $f'(0) = 0$ and $f''(\theta) > 0$ for $\theta > 0$ implies that $f'(\theta) > 0$ for $\theta > 0$. Combining this with $f(0) = 0$, we have $f(\theta) > 0$ for $\theta > 0$. Thus $\cos\theta > 1 - \dfrac{\theta^2}{2}$ for $\theta > 0$. On the other hand, let $g(\theta) = 1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24} - \cos\theta$. Then

$$\begin{aligned} g'(\theta) &= -\theta + \frac{\theta^3}{6} + \sin\theta \\ g''(\theta) &= -1 + \frac{\theta^2}{2} + \cos\theta \end{aligned}$$

Now $g'(0) = 0$ and we have proved that $g''(\theta) = \cos\theta - 1 + \dfrac{\theta^2}{2} > 0$ for any $\theta > 0$. Thus $g'(\theta) > 0$ for any $\theta > 0$. Combining this with $g(0) = 0$, we see that $g(\theta) > 0$ for any $\theta > 0$. Therefore $1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24} > \cos\theta$ for $\theta > 0$.

5. Let $f(t) = \dfrac{a_1 t^2}{2} + \dfrac{a_2 t^3}{3} + \cdots + \dfrac{a_n t^{n+1}}{n+1}$. Then

$$f'(t) = a_1 t + a_2 t^2 + \cdots + a_n t^n.$$

By mean value theorem, there exists $0 < x < 1$ such that

$$\begin{aligned} f'(x) &= \frac{f(1) - f(0)}{1 - 0} \\ a_1 x + a_2 x^2 + \cdots + a_n x^n &= \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1}. \end{aligned}$$

6. Suppose $f(a) = f(b) = 0$.

   (a) Let $h(x) = e^x f(x)$ which is continuous on $[a, b]$ and differentiable on (a, b). Then $h(a) = h(b) = 0$ and $h'(x) = e^x(f'(x) + f(x))$. By Rolle's theorem, there exists $a < \xi < b$ such that

   $$h'(\xi) = e^{\xi}(f'(\xi) + f(\xi)) = 0$$

   which implies $f'(\xi) + f(\xi) = 0$ since $e^{\xi} > 0$.

(b) Let $p(x) = e^{g(x)}f(x)$ which is continuous on $[a,b]$ and differentiable on (a,b). Then $p(a) = p(b) = 0$ and $p'(x) = e^{g(x)}(f'(x) + g'(x)f(x))$. By Rolle's theorem, there exists $a < \eta < b$ such that

$$p'(\eta) = e^{g(\eta)}(f'(\eta) + g'(\eta)f(\eta)) = 0$$

which implies $f'(\eta) + g'(\eta)f(\eta) = 0$ since $e^{g(\eta)} > 0$.

7. Let $g(x) = e^{-x}f(x) - f(0)$. Then $g(0) = 0$ and

$$g'(x) = -e^{-x}f(x) + e^{-x}f'(x) \leq -e^{-x}f(x) + e^{-x}f(x) = 0$$

for any $x > 0$. Thus $g(x) = e^{-x}f(x) - f(0) \leq 0$ for any $x > 0$ which implies $f(x) \leq f(0)e^x$ for any $x > 0$.

8. (a) Note that

$$\begin{aligned} F(a) &= f(a) - \frac{(a-b)(a-c)}{(a-b)(a-c)}f(a) = 0 \\ F(b) &= f(b) - \frac{(b-c)(b-a)}{(b-c)(b-a)}f(b) = 0. \end{aligned}$$

By Rolle's theorem, there exists $\zeta \in (a,b)$ such that $F'(\zeta) = 0$.

(b) Note that

$$\begin{aligned} & F'(x) \\ = \ & f'(x) - \frac{(2x-(b+c))f(a)}{(a-b)(a-c)} - \frac{(2x-(c+a))f(b)}{(b-c)(b-a)} - \frac{(2x-(a+b))f(c)}{(c-b)(c-a)} \\ & F''(x) \\ = \ & f''(x) - \frac{2f(a)}{(a-b)(a-c)} - \frac{2f(b)}{(b-c)(b-a)} - \frac{2f(c)}{(c-b)(c-a)} \end{aligned}$$

By (a), there exists $\zeta \in (a,b)$ such that $F'(\zeta) = 0$. Similarly, there exists $\xi \in (b,c)$ such that $F'(\xi) = 0$. By applying Rolle's theorem to $F'(x)$, there exists $a < \zeta < \eta < \xi < b$ such that

$$F''(\eta) = 0$$

$$f''(\eta) - \frac{2f(a)}{(a-b)(a-c)} - \frac{2f(b)}{(b-c)(b-a)} - \frac{2f(c)}{(c-b)(c-a)} = 0$$

$$\frac{f''(\eta)}{2} = \frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-b)(c-a)}$$

9. (a) Let $f(x) = x \ln x$ for $x > 0$. By mean value theorem, there exists $\xi \in (a, b)$ such that

$$f'(\xi)\frac{f(b) - f(a)}{b - a}$$

Since $f'(x) = 1 + \ln x$ is strictly increasing, we have

$$\begin{array}{ccccc} f'(a) & < & f'(\xi) & < & f'(b) \\ f'(a) & < & \dfrac{f(b) - f(a)}{b - a} & < & f'(b) \\ 1 + \ln a & < & \dfrac{b \ln b - a \ln a}{b - a} & < & 1 + \ln b \\ (1 + \ln a)(b - a) & < & b \ln b - a \ln a & < & (1 + \ln b)(b - a) \end{array}$$

(b) Let $f(x) = xe^{\frac{1}{x}}$. Then

$$f'(x) = e^{\frac{1}{x}} + xe^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = \left(1 - \frac{1}{x}\right)e^{\frac{1}{x}}$$

Applying mean value theorem to $f(x)$ on $(\frac{1}{b}, \frac{1}{a})$, there exists $a < c < b$ such that

$$\begin{aligned} \frac{f(\frac{1}{a}) - f(\frac{1}{b})}{\frac{1}{a} - \frac{1}{b}} &= f'(\frac{1}{c}) \\ \frac{\frac{e^a}{a} - \frac{e^b}{b}}{\frac{1}{a} - \frac{1}{b}} &= (1 - c)e^c \\ \frac{be^a - ae^b}{b - a} &= (1 - c)e^c \\ ae^b - be^a &= (b - a)(c - 1)e^c \end{aligned}$$

10. (a) For any x, by applying mean value theorem to $f(x)$ on $(x, x + 1)$, there exists $x < \xi < x + 1$ such that

$$f'(\xi) = \frac{f(x + 1) - f(x)}{x + 1 - x} = f(x + 1) - f(x)$$

Since $f'(x) < 0$ for any x, $f'(x)$ is strictly decreasing. Thus $f'(x + 1) < f'(\xi) < f'(x)$ which implies $f'(x + 1) < f(x + 1) - f(x) < f'(x)$.

(b) By (a),

$$\begin{aligned} f'(1) &< f(1) - f(0) < f'(0) \\ f'(2) &< f(2) - f(1) < f'(1) \\ f'(3) &< f(3) - f(2) < f'(2) \end{aligned}$$

Adding up the above inequalities, we have

$$f'(1) + f'(2) + f'(3) < f(3) - f(0) < f'(0) + f'(1) + f'(2).$$

11. Observe that at least one of the following holds. Either λ lies between $f'(a)$ and $\dfrac{f(b)-f(a)}{b-a}$, or λ lies between $f'(b)$ and $\dfrac{f(b)-f(a)}{b-a}$. Suppose λ lies between $f'(a)$ and $\dfrac{f(b)-f(a)}{b-a}$. Define

$$f_a(t) = \begin{cases} f'(a) & \text{if } t = a \\ \dfrac{f(t)-f(a)}{t-a} & \text{if } t \neq a \end{cases}$$

Then $f_a(t)$ is continuous on $[a, b]$ since

$$\lim_{t\to 0} f_a(t) = \lim_{t\to 0} \frac{f(t)-f(a)}{t-a} = f'(a) = f_a(a).$$

Also $f_a(t)$ is differentiable at any $t \in (a, b)$ because $f(x)$ is differentiable at any $x \in (a, b)$. Since λ lies between $f_a(a) = f'(a)$ and $f_a(b) = \dfrac{f(b)-f(a)}{b-a}$, by applying intermediate value theorem to $f_a(t)$ on $[a, b]$, there exists $\eta \in [a, b]$ such that $f_a(\eta) = \dfrac{f(\eta)-f(a)}{\eta-a} = \lambda$. Applying mean value theorem to $f(x)$ on $[a, \eta]$, there exists $\xi \in (a, \eta)$ such that $f'(\xi) = \dfrac{f(\eta)-f(a)}{\eta-a} = \lambda$.

Suppose λ lies between $f'(b)$ and $\dfrac{f(b)-f(a)}{b-a}$. Define

$$f_b(t) = \begin{cases} f'(b) & \text{if } t = b \\ \frac{f(b)-f(t)}{b-t} & \text{if } t \neq b \end{cases}$$

The rest of the argument is more or less the same.

12. (a) (i)

$$x = \frac{x}{p} + \frac{x}{q} < \frac{x}{p} + \frac{y}{q} < \frac{y}{p} + \frac{y}{q} = y.$$

(ii) Applying mean value theorem to $f(x)$ on (x, z), there exists $x < \xi < z$ such that

$$f'(\xi) = \frac{f(z) - f(x)}{z - x}.$$

Now

$$z - x = \frac{x}{p} + \frac{y}{q} - \left(\frac{x}{p} + \frac{x}{q}\right) = \frac{y - x}{q}.$$

Hence

$$f'(\xi) = \frac{q(f(z) - f(x))}{y - x}.$$

(iii) Applying mean value theorem to $f(x)$ on (z, y), there exists $z < \eta < y$ such that

$$f'(\eta) = \frac{p(f(y) - f(z))}{y - x}.$$

Since $f''(t) \geq 0$ for any t and $\xi < \eta$, we have $f'(\xi) \leq f'(\eta)$ and therefore

$$\begin{aligned} \frac{q(f(z) - f(x))}{y - x} &\leq \frac{p(f(y) - f(z))}{y - x} \\ (p+q)f(z) &\leq qf(x) + pf(y) \\ f(z) &\leq \frac{qf(x)}{p+q} + \frac{pf(y)}{p+q} \\ &= \frac{f(x)}{p} + \frac{f(y)}{q} \end{aligned}$$

(b) (i) Without loss of generality, we may assume $p \ln a < q \ln b$. Take $x = p \ln a$, $y = q \ln b$ and

$$z = \frac{x}{p} + \frac{y}{q} = \ln a + \ln b = \ln(ab).$$

By (a)(iii), we have

$$f(\ln(ab)) \leq \frac{f(p \ln a)}{p} + \frac{f(q \ln b)}{q}.$$

(ii) Let $f(t) = e^t$. Then $f''(t) = e^t > 0$ for any $t \in \mathbb{R}$. By (b)(i), we have

$$\begin{aligned} e^{\ln(ab)} &\leq \frac{e^{p\ln a}}{p} + \frac{e^{q\ln b}}{q} \\ ab &\leq \frac{a^p}{p} + \frac{b^q}{q}. \end{aligned}$$

13. (a) Consider the function $g(x)$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

Then $\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} f(x) = 0 = g(0)$. Thus $g(x)$ is continuous on $[0, +\infty)$. Moreover, $g'(x) = f'(x) > 0$ for any $x > 0$. For any $x > 0$, applying mean value theorem to $g(x)$ on $[0, x]$, there exist $0 < \xi < x$ such that

$$\begin{aligned} \frac{g(x) - g(0)}{x - 0} &= g'(\xi) \\ \frac{f(x) - 0}{x} &= f'(\xi) \end{aligned}$$

which implies $f(x) = f'(\xi)x > 0$.

(b) Consider $f(x) = (1 + x)\ln(1 + x) - x\ln x$. Then

$$f'(x) = \ln(1 + x) + 1 - \ln x - 1 = \ln(1 + x) - \ln x > 0$$

for any $x > 0$ and

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} (1 + x)\ln(1 + x) - x\ln x \\ &= -\lim_{x\to 0^+} x\ln x \\ &= -\lim_{t\to +\infty} \frac{\ln(\frac{1}{t})}{t} \\ &= \lim_{t\to +\infty} \frac{\ln t}{t} \\ &= 0. \end{aligned}$$

By (a), we have $f(x) = (1+x)\ln(1+x) - x\ln x > 0$ for any $x > 0$.

(c) Let $a > 0$ and $g(t) = \dfrac{\ln(1+a^t)}{t}$. Then for any $t > 0$,

$$\begin{aligned} g'(t) &= \frac{\frac{t}{1+a^t} \cdot a^t \ln a - \ln(1+a^t)}{t^2} \\ &= \frac{a^t \ln(a^t) - (1+a^t)\ln(1+a^t)}{t^2(1+a^t)} \\ &< 0 \end{aligned}$$

where the last inequality follows from (b) since $a^t > 0$ for any $t > 0$.

(d) Let $a = \dfrac{v}{u}$. Note that $g(x)$ is continuous on $[p, q]$ and differentiable on (p, q). By the mean value theorem, there exists $\xi \in (p, q)$ such that

$$\begin{aligned} \frac{g(q) - g(p)}{q - p} &= g'(\xi) < 0 \\ \frac{\ln(1+a^q)}{q} - \frac{\ln(1+a^p)}{p} &< 0 \\ \ln(1+a^q)^{\frac{1}{q}} &< \ln(1+a^p)^{\frac{1}{p}} \\ (1+a^q)^{\frac{1}{q}} &< (1+a^p)^{\frac{1}{p}} \\ \left(1 + \frac{v^q}{u^q}\right)^{\frac{1}{q}} &< \left(1 + \frac{v^p}{u^p}\right)^{\frac{1}{p}} \\ (u^q + v^q)^{\frac{1}{q}} &< (u^p + v^p)^{\frac{1}{p}}. \end{aligned}$$