Linear Systems and Matrices

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1. Linear Systems and Matrices
   - Row Echelon Form
   - Matrix Operations
   - Inverse of matrices
   - Determinants
   - Linear Equations and Curve Fitting

System of m linear equations in n unknowns (linear system)

$$\left\{\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & = & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}\right. .$$

Matrix form

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} .$$

Augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

Definition

An **elementary row operation** *is an operation on a matrix of one of the following form.*

1. *Multiply a row by a non-zero constant.*
2. *Interchange two rows.*
3. *Replace a row by its sum with a multiple of another row.*

Definition

Two matrices A and B are said to be **row equivalent** *if we can use elementary row operations to get B from A.*

Proposition

If the augmented matrices of two linear systems are row equivalent, then the two systems are equivalent, i.e., they have the same solution set.

Definition

A matrix E is said to be in **row echelon form** *if*

1. *The first nonzero entry of each row of E is 1.*
2. *Every row of E that consists entirely of zeros lies beneath every row that contains a nonzero entry.*
3. *In each row of E that contains a nonzero entry, the number of leading zeros is strictly less than that in the preceding row.*

Proposition

Any matrix can be transformed into row echelon form by elementary row operations. This process is called **Gaussian elimination**.

Row echelon form of augmented matrix.

- Those variables that correspond to columns containing leading entries are called **leading variables**
- All the other variables are called **free variables**.

A system in row echelon form can be solved easily by **back substitution**.

Example

Solve the linear system

$$\left\{\begin{array}{rcrcrcr} x_1 & + & x_2 & - & x_3 & = & 5 \\ 2x_1 & - & x_2 & + & 4x_3 & = & -2 \\ x_1 & - & 2x_2 & + & 5x_3 & = & -4 \end{array}\right. .$$

Solution:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 5 \\ 2 & -1 & 4 & -2 \\ 1 & -2 & 5 & -4 \end{array}\right) \xrightarrow{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1}} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & -3 & 6 & -12 \\ 0 & -3 & 6 & -9 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{1}{3} R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & -3 & 6 & -9 \end{array}\right) \xrightarrow{R_3 \to R_3 + 3R_2} \left(\begin{array}{ccc|c} 1 & 1 & -2 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

The third row of the last matrix corresponds to the equation

$$0 = 3$$

which is absurd. Therefore the solution set is empty and the system is inconsistent. $\square$

Example

Solve the linear system

$$\left\{\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 2 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 2x_5 & = & 3 \\ x_1 & + & x_2 & + & x_3 & + & 2x_4 & + & 3x_5 & = & 2 \end{array}\right. .$$

Solution:

$$\left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{array}\right) \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - R_2} \left(\begin{array}{ccccc|c} 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Thus the system is equivalent to the following system

$$\left\{\begin{array}{ccccccccccc} x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & = & 2 \\ & & & & & & x_4 & + & x_5 & = & 1 \\ & & & & & & & & x_5 & = & -1 \end{array}\right. .$$

The solution of the system is

$$\begin{cases} x_5 = -1 \\ x_4 = 1 - x_5 = 2 \\ x_1 = 2 - x_2 - x_3 - x_4 - x_5 = 1 - x_2 - x_3 \end{cases}$$

Here x_1, x_4, x_5 are leading variables while x_2, x_3 are free variables. Another way of expressing the solution is

$$(x_1, x_2, x_3, x_4, x_5) = (1 - \alpha - \beta, \alpha, \beta, 2, -1),\ \alpha, \beta \in \mathbb{R}.$$

Definition

A matrix E is said to be in **reduced row echelon form** *(or E is a* **reduced row echelon matrix***) if it satisfies all the following properties:*

1. *It is in row echelon form.*
2. *Each leading entry of E is the only nonzero entry in its column.*

Proposition

Every matrix is row equivalent to one and only one matrix in reduced row echelon form.

Example

Find the reduced row echelon form of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{pmatrix}.$$

Solution:

$$\begin{pmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{pmatrix} \xrightarrow{\begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 2R_1 \end{array}} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{pmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 7 & 15 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 2R_2} \begin{pmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{pmatrix} \xrightarrow{\begin{array}{l} R_1 \to R_1 + 3R_3 \\ R_2 \to R_2 - 2R_3 \end{array}} \begin{pmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{pmatrix}$$

Example

Solve the linear system

$$\left\{\begin{array}{ccccccccc} x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 5 \\ x_1 & + & 2x_2 & + & 2x_3 & + & 3x_4 & = & 4 \\ x_1 & + & 2x_2 & + & x_3 & + & 2x_4 & = & 3 \end{array}\right. .$$

Solution:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 1 & 2 & 3 \end{pmatrix} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & -1 & -1 & -1 \\ 0 & 0 & -2 & -2 & -2 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \end{pmatrix} \xrightarrow{R_3 \to R_3 + 2R_2} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{pmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Now x_1, x_3 are leading variables while x_2, x_4 are free variables. The solution of the system is

$$(x_1, x_2, x_3, x_4) = (2 - 2\alpha - \beta, \alpha, 1 - \beta, \beta), \ \alpha, \beta \in \mathbb{R}.$$

$\square$

Theorem

Let

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

be a linear system, where $\mathbf{A}$ *is an* $m \times n$ *matrix. Let* $\mathbf{R}$ *be the unique* $m \times (n+1)$ *reduced row echelon matrix of the augmented matrix* $(\mathbf{A}|\mathbf{b})$*. Then the system has*

1. *no solution if the last column of* $\mathbf{R}$ *contains a leading entry.*
2. *unique solution if (1) does not holds and all variables are leading variables.*
3. *infinitely many solutions if (1) does not holds and there exists at least one free variables.*

Theorem

Let $\mathbf{A}$ *be an* $n \times n$ *matrix. Then homogeneous linear system*

$$\mathbf{Ax} = \mathbf{0}$$

with coefficient matrix $\mathbf{A}$ *has only trivial solution if and only if* $\mathbf{A}$ *is row equivalent to the identity matrix* $\mathbf{I}$.

Definition

We define the following operations for matrices.

1. **Addition**: *Let* $\mathbf{A} = [a_{ij}]$ *and* $\mathbf{B} = [b_{ij}]$ *be two* $m \times n$ *matrices. Define*

$$[\mathbf{A}+\mathbf{B}]_{ij} = a_{ij} + b_{ij}.$$

That is

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}.$$

Definition

2 **Scalar multiplication**: *Let* $\mathbf{A} = [a_{ij}]$ *be an* $m \times n$ *matrix and* c *be a scalar. Then*

$$[c\mathbf{A}]_{ij} = ca_{ij}.$$

That is

$$c \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{pmatrix}.$$

Definition

3 **Matrix multiplication**: *Let* $\mathbf{A} = [a_{ij}]$ *be an* $m \times n$ *matrix and* $\mathbf{B} = [b_{jk}]$ *be an* $n \times r$. *Then their matrix product* $\mathbf{AB}$ *is an* $m \times r$ *matrix where*

$$[\mathbf{AB}]_{ik} = \sum_{j=1}^{n} a_{ij} b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + \cdots + a_{in} b_{nk}.$$

For example: If $\mathbf{A}$ *is a* 3×2 *matrix and* $\mathbf{B}$ *is a* 2×2 *matrix, then*

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \\ a_{31} b_{11} + a_{32} b_{21} & a_{31} b_{12} + a_{32} b_{22} \end{pmatrix}$$

is a 3×2 *matrix.*

1. A **zero matrix**, denoted by $\mathbf{0}$, is a matrix whose entries are all zeros.
2. An **identity matrix**, denoted by $\mathbf{I}$, is a square matrix that has ones on its principal diagonal and zero elsewhere.

Theorem (Properties of matrix algebra)

Let $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *be matrices of appropriate sizes to make the indicated operations possible and* a, b *be real numbers, then following identities hold.*

1. $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$
2. $\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$
3. $\mathbf{A} + \mathbf{0} = \mathbf{0} + \mathbf{A} = \mathbf{A}$
4. $a(\mathbf{A} + \mathbf{B}) = a\mathbf{A} + a\mathbf{B}$
5. $(a + b)\mathbf{A} = a\mathbf{A} + b\mathbf{A}$
6. $a(b\mathbf{A}) = (ab)\mathbf{A}$
7. $a(\mathbf{AB}) = (a\mathbf{A})\mathbf{B} = \mathbf{A}(a\mathbf{B})$
8. $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
9. $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$
10. $(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$
11. $\mathbf{A0} = \mathbf{0A} = \mathbf{0}$
12. $\mathbf{AI} = \mathbf{IA} = \mathbf{A}$

Proof.

We only prove (8) and the rest are obvious. Let $\mathbf{A} = [a_{ij}]$ be $m \times n$, $\mathbf{B} = [b_{jk}]$ be $n \times r$ and $\mathbf{C} = [c_{kl}]$ be $r \times s$ matrices. Then

$$\begin{aligned}
[(\mathbf{AB})\mathbf{C}]_{il} &= \sum_{k=1}^{r} [\mathbf{AB}]_{ik} c_{kl} \\
&= \sum_{k=1}^{r} \left(\sum_{j=1}^{n} a_{ij} b_{jk} \right) c_{kl} \\
&= \sum_{j=1}^{n} a_{ij} \left(\sum_{k=1}^{r} b_{jk} c_{kl} \right) \\
&= \sum_{j=1}^{n} a_{ij} [\mathbf{BC}]_{jl} \\
&= [\mathbf{A}(\mathbf{BC})]_{il}
\end{aligned}$$

$\square$

Remarks:

1. In general, $\mathbf{AB} \neq \mathbf{BA}$. For example:

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Then

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$$

$$\mathbf{BA} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

2. $\mathbf{AB} = \mathbf{0}$ does not implies that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. For example:

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0} \text{ and } \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \neq \mathbf{0}$$

But

$$\mathbf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Definition

Let $\mathbf{A} = [a_{ij}]$ *be an* $m \times n$ *matrix. Then the* **transpose** *of* $\mathbf{A}$ *is the* $n \times m$ *matrix defined by interchanging rows and columns and is denoted by* $\mathbf{A}^T$*, i.e.,*

$$[\mathbf{A}^T]_{ji} = a_{ij} \text{ for } 1 \leq j \leq n, 1 \leq i \leq m.$$

Example

1. $\begin{pmatrix} 2 & 0 & 5 \\ 4 & -1 & 7 \end{pmatrix}^T = \begin{pmatrix} 2 & 4 \\ 0 & -1 \\ 5 & 7 \end{pmatrix}$
2. $\begin{pmatrix} 7 & -2 & 6 \\ 1 & 2 & 3 \\ 5 & 0 & 4 \end{pmatrix}^T = \begin{pmatrix} 7 & 1 & 5 \\ -2 & 2 & 0 \\ 6 & 3 & 4 \end{pmatrix}$

Theorem (Properties of transpose)

For any $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$,

1. $(\mathbf{A}^T)^T = \mathbf{A}$;
2. $(\mathbf{A}+\mathbf{B})^T = \mathbf{A}^T + \mathbf{B}^T$;
3. $(c\mathbf{A})^T = c\mathbf{A}^T$;
4. $(\mathbf{AB})^T = \mathbf{B}^T\mathbf{A}^T$.

Definition

A square matrix **A** *is said to be* **invertible**, *if there exists a matrix* **B** *such that*

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}.$$

We say that **B** *is a (multiplicative) inverse of A.*

Theorem

If $\mathbf{A}$ is invertible, then the inverse of $\mathbf{A}$ is unique.

Proof.

Suppose $\mathbf{B}_1$ and $\mathbf{B}_2$ are multiplicative inverses of $\mathbf{A}$. Then

$$\mathbf{B}_2 = \mathbf{B}_2\mathbf{I} = \mathbf{B}_2(\mathbf{A}\mathbf{B}_1) = (\mathbf{B}_2\mathbf{A})\mathbf{B}_1 = \mathbf{I}\mathbf{B}_1 = \mathbf{B}_1.$$

□

The unique inverse of $\mathbf{A}$ is denoted by $\mathbf{A}^{-1}$.

Proposition

The 2×2 *matrix*

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is invertible if and only if $ad - bc \neq 0$, *in which case*

$$\mathbf{A}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proposition

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be two invertible* $n \times n$ *matrices.*

1. $\mathbf{A}^{-1}$ *is invertible and* $(\mathbf{A}^{-1})^{-1} = \mathbf{A}$;
2. *For any nonnegative integer* k, $\mathbf{A}^k$ *is invertible and* $(\mathbf{A}^k)^{-1} = (\mathbf{A}^{-1})^k$;
3. *The product* $\mathbf{AB}$ *is invertible and*
$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1};$$
4. $\mathbf{A}^T$ *is invertible and*
$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T.$$

Proof.

We prove (3) only.

$$\begin{aligned}(\mathbf{AB})(\mathbf{B}^{-1}\mathbf{A}^{-1}) &= \mathbf{A}(\mathbf{BB}^{-1})\mathbf{A}^{-1} = \mathbf{AIA}^{-1} = \mathbf{AA}^{-1} = \mathbf{I}\\ (\mathbf{B}^{-1}\mathbf{A}^{-1})(\mathbf{AB}) &= \mathbf{B}^{-1}(\mathbf{A}^{-1}\mathbf{A})\mathbf{B} = \mathbf{B}^{-1}\mathbf{IB} = \mathbf{B}^{-1}\mathbf{B} = \mathbf{I}\end{aligned}$$

Therefore $\mathbf{AB}$ is invertible and $\mathbf{B}^{-1}\mathbf{A}^{-1}$ is the inverse of $\mathbf{AB}$. □

Theorem

If the $n \times n$ matrix $\mathbf{A}$ is invertible, then for any n-vector $\mathbf{b}$ the system $\mathbf{Ax} = \mathbf{b}$ has the unique solution $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$.

Example

Solve the system

$$\left\{ \begin{array}{rcrcr} 4x_1 & + & 6x_2 & = & 6 \\ 5x_1 & + & 9x_2 & = & 18 \end{array} \right. .$$

Solution: Let $\mathbf{A} = \begin{pmatrix} 4 & 6 \\ 5 & 9 \end{pmatrix}$. Then

$$\mathbf{A}^{-1} = \frac{1}{(4)(9) - (5)(6)} \begin{pmatrix} 9 & -6 \\ -5 & 4 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{pmatrix}$$

Thus the solution is

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} 6 \\ 18 \end{pmatrix} = \begin{pmatrix} -9 \\ 7 \end{pmatrix}$$

Therefore $(x_1, x_2) = (-9, 7)$. $\square$

Definition

A square matrix $\mathbf{E}$ *is called an* **elementary matrix** *if it can be obtained by performing a single elementary row operation on* $\mathbf{I}$.

Proposition

Let $\mathbf{E}$ *be the elementary matrix obtained by performing a certain elementary row operation on* $\mathbf{I}$. *Then the result of performing the same elementary row operation on a matrix* $\mathbf{A}$ *is* $\mathbf{EA}$.

Proposition

Every elementary matrix is invertible.

Example

Examples of elementary matrices associated to elementary row operations and their inverses.

Elementary row operation	*Interchanging two rows*	*Multiplying a row by a nonzero constant*	*Adding a multiple of a row to another row*
Elementary matrix	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$
Inverse	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Theorem

Let $\mathbf{A}$ *be a square matrix. Then the following statements are equivalent.*

1. $\mathbf{A}$ *is invertible*
2. $\mathbf{A}$ *is row equivalent to* $\mathbf{I}$
3. $\mathbf{A}$ *is a product of elementary matrices*

Proof.

It follows easily from the fact that an $n \times n$ reduced row echelon matrix is invertible if and only if it is the identity matrix $\mathbf{I}$. $\square$

Let $\mathbf{A}$ be an invertible matrix. Then the above theorem tells us that there exists elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_k$ such that

$$\mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}.$$

Multiplying both sides by $(\mathbf{E}_1)^{-1}(\mathbf{E}_2)^{-1} \cdots (\mathbf{E}_{k-1})^{-1}(\mathbf{E}_k)^{-1}$ we have

$$\mathbf{A} = (\mathbf{E}_1)^{-1}(\mathbf{E}_2)^{-1} \cdots (\mathbf{E}_{k-1})^{-1}(\mathbf{E}_k)^{-1}.$$

Therefore

$$\mathbf{A}^{-1} = \mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_2 \mathbf{E}_1$$

by Proposition 3.4.

Theorem

Let $\mathbf{A}$ *be a square matrix. Suppose we can preform elementary row operation to the augmented matrix* $(\mathbf{A}|\mathbf{I})$ *to obtain a matrix of the form* $(\mathbf{I}|\mathbf{E})$*, then* $\mathbf{A}^{-1} = \mathbf{E}$*.*

Proof.

Let $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_k$ be elementary matrices such that

$$\mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_2 \mathbf{E}_1 (\mathbf{A}|\mathbf{I}) = (\mathbf{I}|\mathbf{E}).$$

Then the multiplication on the left submatrix gives

$$\mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{A} = \mathbf{I}$$

and the multiplication of the right submatrix gives

$$\mathbf{E} = \mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_2 \mathbf{E}_1 \mathbf{I} = \mathbf{A}^{-1}.$$

Example

Find the inverse of

$$\begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{pmatrix}$$

Solution:

$$\left(\begin{array}{ccc|ccc} 4 & 3 & 2 & 1 & 0 & 0 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 - R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 5 & 6 & 3 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{\substack{R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 3R_1}} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 16 & 3 & -5 & 1 & 5 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 11 & 2 & -3 & 0 & 4 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 5 & 1 & -2 & 1 & 1 \\ 0 & 1 & 0 & 1 & -2 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 5 & 1 & -2 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 - 5R_2} \left(\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right)$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -4 & 3 \\ 0 & 1 & 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -7 & 11 & -9 \end{array}\right)$$

Therefore

$$\mathbf{A}^{-1} = \begin{pmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{pmatrix}.$$

Example

Find a 3×2 *matrix* $\mathbf{X}$ *such that*

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 1 & 3 & 4 \end{pmatrix} \mathbf{X} = \begin{pmatrix} 0 & -3 \\ -1 & 4 \\ 2 & 1 \end{pmatrix}.$$

Solution:

$$\left(\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & -3 \\ 2 & 1 & 2 & -1 & 4 \\ 1 & 3 & 4 & 2 & 1 \end{array}\right)$$

$$\begin{array}{c} R_2 \to R_2 - 5R_1 \\ R_3 \to R_3 - 3R_1 \\ \longrightarrow \end{array} \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & -3 \\ 0 & -3 & -4 & -1 & 10 \\ 0 & 1 & 1 & 2 & 4 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & -3 & -4 & -1 & 10 \end{array}\right)$$

$$\xrightarrow{R_3\to R_3+3R_2}\left(\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & -1 & 5 & 22 \end{array}\right)$$

$$\xrightarrow{R_3\to -R_3}\left(\begin{array}{ccc|cc} 1 & 2 & 3 & 0 & -3 \\ 0 & 1 & 1 & 2 & 4 \\ 0 & 0 & 1 & -5 & -22 \end{array}\right)$$

$$\xrightarrow{\substack{R_1\to R_1-3R_3 \\ R_2\to R_2-R_3}}\left(\begin{array}{ccc|cc} 1 & 2 & 0 & 15 & 63 \\ 0 & 1 & 0 & 7 & 26 \\ 0 & 5 & 1 & -5 & -22 \end{array}\right)$$

$$\xrightarrow{R_1\to R_1-2R_2}\left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 11 \\ 0 & 1 & 0 & 7 & 26 \\ 0 & 0 & 1 & -5 & -22 \end{array}\right)$$

Therefore we may take

$$\mathbf{X}=\begin{pmatrix} 1 & 11 \\ 7 & 26 \\ -5 & -22 \end{pmatrix}.$$

$\square$

Definition

Let $\mathbf{A} = [a_{ij}]$ be an $n \times n$ matrix.

1. *The ij-th* **minor** *of* $\mathbf{A}$ *is the determinant M_{ij} of the $(n-1) \times (n-1)$ submatrix that remains after deleting the i-th row and the j-th column of* $\mathbf{A}$.
2. *The ij-th* **cofactor** *of* $\mathbf{A}$ *is defined by*

$$A_{ij} = (-1)^{i+j} M_{ij}.$$

Definition

Let $\mathbf{A} = [a_{ij}]$ *be an* $n \times n$ *matrix. The* **determinant** $\det(\mathbf{A})$ *of* $\mathbf{A}$ *is defined inductively as follow.*

1. *If* $n = 1$, *then* $\det(\mathbf{A}) = a_{11}$.
2. *If* $n > 1$, *then*

$$\det(\mathbf{A}) = \sum_{k=1}^{n} a_{1k}A_{1k} = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n},$$

where A_{ij} *is the ij-th cofactor of* $\mathbf{A}$.

Example

When $n = 1, 2$ or 3, we have the following.

1. *The determinant of a 1×1 matrix is*

$$\begin{vmatrix} a_{11} \end{vmatrix} = a_{11}$$

2. *The determinant of a 2×2 matrix is*

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

3. *The determinant of a 3×3 matrix is*

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$$\begin{aligned}
&\begin{vmatrix} 4 & 3 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{vmatrix} \\
=\ & 4\begin{vmatrix} 2 & 0 & 1 \\ 0 & 0 & 3 \\ 1 & 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 3 & 0 & 1 \\ 1 & 0 & 3 \\ 0 & 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 3 & 2 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{vmatrix} \\
=\ & 4\left(2\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix}\right) \\
& -3\left(3\begin{vmatrix} 0 & 3 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}\right) \\
& -\left(3\begin{vmatrix} 0 & 0 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} + 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right) \\
=\ & 4\,(2(-6)) - 3\,(3(-6) + 1(2)) - (-2(2)) \\
=\ & 4
\end{aligned}$$

$\square$

Theorem

Let $\mathbf{A} = [a_{ij}]$ *be an* $n \times n$ *matrix. Then*

$$\det(\mathbf{A}) = \sum_{\sigma \in S_n} sign(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where S_n *is the set of all permutations of* $\{1, 2, \cdots, n\}$ *and*

$$sign(\sigma) = \begin{cases} 1 & \text{if } \sigma \text{ is an even permutation,} \\ -1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

1. There are $n!$ number of terms for an $n \times n$ determinant.
2. Here we write down the $4! = 24$ terms of a 4×4 determinant.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$= \begin{array}{l} a_{11}a_{22}a_{33}a_{44} - a_{11}a_{22}a_{34}a_{43} - a_{11}a_{23}a_{32}a_{44} + a_{11}a_{23}a_{34}a_{42} \\ +a_{11}a_{24}a_{32}a_{43} - a_{11}a_{24}a_{33}a_{42} - a_{12}a_{21}a_{33}a_{44} + a_{12}a_{21}a_{34}a_{43} \\ +a_{12}a_{23}a_{31}a_{44} - a_{12}a_{23}a_{34}a_{41} - a_{12}a_{24}a_{31}a_{43} + a_{12}a_{24}a_{33}a_{41} \\ +a_{13}a_{21}a_{32}a_{44} - a_{13}a_{21}a_{34}a_{42} - a_{13}a_{22}a_{31}a_{44} + a_{13}a_{22}a_{34}a_{41} \\ +a_{13}a_{24}a_{31}a_{42} - a_{13}a_{24}a_{32}a_{41} - a_{14}a_{21}a_{32}a_{43} + a_{14}a_{21}a_{33}a_{42} \\ +a_{14}a_{22}a_{31}a_{43} - a_{14}a_{22}a_{33}a_{41} - a_{14}a_{23}a_{31}a_{42} + a_{14}a_{23}a_{32}a_{41} \end{array}$$

Theorem

The determinant of an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ can be obtained by expansion along any row or column, i.e., for any $1 \leq i \leq n$, we have

$$\det(\mathbf{A}) = a_{i1}A_{il} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$

and for any $1 \leq j \leq n$, we have

$$\det(\mathbf{A}) = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}.$$

Example

We can expand the determinant along the third column.

$$\begin{aligned}
\begin{vmatrix} 4 & 3 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 0 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{vmatrix} &= -2\begin{vmatrix} 4 & 3 & 1 \\ 3 & 2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \\
&= -2\left(-3\begin{vmatrix} 3 & 1 \\ 1 & 3 \end{vmatrix} + 2\begin{vmatrix} 4 & 1 \\ 1 & 3 \end{vmatrix}\right) \\
&= -2\,(-3(8) + 2(11)) \\
&= 4
\end{aligned}$$

$\square$

Proposition

Properties of determinant.

1. $\det(\mathbf{I}) = 1$;
2. *Suppose that the matrices* $\mathbf{A}_1$, $\mathbf{A}_2$ *and* $\mathbf{B}$ *are identical except for their* i*-th row (or column) and that the* i*-th row (or column) of* $\mathbf{B}$ *is the sum of the* i*-th row (or column) of* $\mathbf{A}_1$ *and* $\mathbf{A}_2$*, then* $\det(\mathbf{B}) = \det(\mathbf{A}_1) + \det(\mathbf{A}_2)$;
3. *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by multiplying a single row (or column) of* $\mathbf{A}$ *by the constant* k*, then* $\det(\mathbf{B}) = k\det(\mathbf{A})$;
4. *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by interchanging two rows (or columns), then* $\det(\mathbf{B}) = -\det(\mathbf{A})$;

Proposition

5. *If* $\mathbf{B}$ *is obtained from* $\mathbf{A}$ *by adding a constant multiple of one row (or column) of* $\mathbf{A}$ *to another row (or column) of* $\mathbf{A}$*, then* $\det(\mathbf{B}) = \det(\mathbf{A})$*;*
6. *If two rows (or columns) of* $\mathbf{A}$ *are identical, then* $\det(\mathbf{A}) = 0$*;*
7. *If* $\mathbf{A}$ *has a row (or column) consisting entirely of zeros, then* $\det(\mathbf{A}) = 0$*;*
8. $\det(\mathbf{A}^T) = \det(\mathbf{A})$*;*
9. *If* $\mathbf{A}$ *is a triangular matrix, then* $\det(\mathbf{A})$ *is the product of the diagonal elements of* $\mathbf{A}$*;*
10. $\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B})$.

Example

$$
\begin{aligned}
\begin{vmatrix} 2 & 2 & 5 & 5 \\ 1 & -2 & 4 & 1 \\ -1 & 2 & -2 & -2 \\ -2 & 7 & -3 & 2 \end{vmatrix}
&= \begin{vmatrix} 0 & 6 & -3 & 3 \\ 1 & -2 & 4 & 1 \\ 0 & 0 & 2 & -1 \\ 0 & 3 & 5 & 4 \end{vmatrix}
\left(\begin{array}{l} R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + R_2 \\ R_4 \to R_4 + 2R_2 \end{array} \right) \\
&= - \begin{vmatrix} 6 & -3 & 3 \\ 0 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} \\
&= -3 \begin{vmatrix} 2 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} \\
&= -3 \left(2 \begin{vmatrix} -1 & 1 \\ 5 & 4 \end{vmatrix} + 3 \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} \right) \\
&= -69
\end{aligned}
$$

□

Example

$$\begin{aligned}
\begin{vmatrix} 2 & 2 & 5 & 5 \\ 1 & -2 & 4 & 1 \\ -1 & 2 & -2 & -2 \\ -2 & 7 & -3 & 2 \end{vmatrix}
&= \begin{vmatrix} 2 & 6 & -3 & 3 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 \\ -2 & 3 & 5 & 4 \end{vmatrix} \begin{pmatrix} C_2 \to C_2 + 2C_1 \\ C_3 \to C_3 - 4C_1 \\ C_4 \to C_4 - C_1 \end{pmatrix} \\
&= -\begin{vmatrix} 6 & -3 & 3 \\ 0 & 2 & -1 \\ 3 & 5 & 4 \end{vmatrix} \\
&= -\begin{vmatrix} 0 & 0 & 3 \\ 2 & 1 & -1 \\ -5 & 9 & 4 \end{vmatrix} \begin{pmatrix} C_1 \to C_1 - 2C_3 \\ C_2 \to C_2 + C_3 \end{pmatrix} \\
&= -3\begin{vmatrix} 2 & 1 \\ -5 & 9 \end{vmatrix} \\
&= -69
\end{aligned}$$

$\square$

Example

Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ *be real numbers and*

$$\mathbf{A} = \begin{pmatrix} 1 & \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ 1 & x & \alpha_2 & \cdots & \alpha_n \\ 1 & \alpha_1 & x & \cdots & \alpha_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \alpha_1 & \alpha_2 & \cdots & x \end{pmatrix}.$$

Show that

$$\det(\mathbf{A}) = (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n).$$

Solution: Note that A is an $(n+1) \times (n+1)$ matrix. For simplicity we assume that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are distinct. Observe that we have the following 3 facts.

1. $\det(\mathbf{A})$ is a polynomial of degree n in x;
2. $\det(\mathbf{A}) = 0$ when $x = \alpha_i$ for some i;
3. The coefficient of x^n of $\det(\mathbf{A})$ is 1.

Then the equality follows by the factor theorem. $\square$

Example

The Vandermonde determinant is defined as

$$V(x_1, x_2, \cdots, x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{vmatrix}.$$

Show that

$$V(x_1, x_2, \cdots, x_n) = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Solution: Using factor theorem, the equality is a consequence of the following 3 facts.

1. $V(x_1, x_2, \cdots, x_n)$ is a polynomial of degree $n(n-1)/2$ in $x_1, x_2, \cdots, x_n$;
2. For any $i \neq j$, $V(x_1, x_2, \cdots, x_n) = 0$ when $x_i = x_j$;
3. The coefficient of $x_2 x_3^2 \cdots x_n^{n-1}$ of $V(x_1, x_2, \cdots, x_n)$ is 1.

$\square$

Lemma

Let $\mathbf{A} = [a_{ij}]$ *be an* $n \times n$ *matrix and* $\mathbf{E}$ *be an* $n \times n$ *elementary matrix. Then*

$$\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}).$$

Definition

Let $\mathbf{A}$ *be a square matrix. We say that* $\mathbf{A}$ *is* **singular** *if the system* $\mathbf{Ax} = \mathbf{0}$ *has non-trivial solution. A square matrix is* **nonsingular** *if it is not singular.*

Theorem

The following properties of an $n \times n$ matrix $\mathbf{A}$ are equivalent.

1. $\mathbf{A}$ *is nonsingular, i.e., the system* $\mathbf{Ax} = \mathbf{0}$ *has only trivial solution* $\mathbf{x} = \mathbf{0}$.
2. $\mathbf{A}$ *is invertible, i.e.,* $\mathbf{A}^{-1}$ *exists.*
3. $\det(\mathbf{A}) \neq 0$.
4. $\mathbf{A}$ *is row equivalent to* $\mathbf{I}$.
5. *For any n-column vector* $\mathbf{b}$*, the system* $\mathbf{Ax} = \mathbf{b}$ *has a unique solution.*
6. *For any n-column vector* $\mathbf{b}$*, the system* $\mathbf{Ax} = \mathbf{b}$ *has a solution.*

Proof.

We prove (3)⇔(4) and leave the rest as an exercise. Multiply elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_k$ to $\mathbf{A}$ so that

$$\mathbf{R} = \mathbf{E}_k \mathbf{E}_{k-1} \cdots \mathbf{E}_1 \mathbf{A}$$

is in reduced row echelon form. Then by the lemma above, we have

$$\det(\mathbf{R}) = \det(\mathbf{E}_k)\det(\mathbf{E}_{k-1})\cdots\det(\mathbf{E}_1)\det(\mathbf{A}).$$

Since determinant of elementary matrices are always nonzero, we have $\det(\mathbf{A})$ is nonzero if and only if $\det(\mathbf{R})$ is nonzero. It is easy to see that the determinant of a reduced row echelon matrix is nonzero if and only if it is the identity matrix $\mathbf{I}$. □

Theorem

Let $\mathbf{A}$ *and* $\mathbf{B}$ *be two* $n \times n$ *matrices. Then*

$$\det(\mathbf{AB}) = \det(\mathbf{A})\det(\mathbf{B}).$$

Proof.

If $\mathbf{A}$ is not invertible, then $\mathbf{AB}$ is not invertible and $\det(\mathbf{AB}) = 0 = \det(\mathbf{A})\det(\mathbf{B})$. If $\mathbf{A}$ is invertible, then there exists elementary matrices $\mathbf{E}_1, \mathbf{E}_2, \cdots, \mathbf{E}_k$ such that $\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1 = \mathbf{A}$. Hence

$$\begin{aligned}
\det(\mathbf{AB}) &= \det(\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1\mathbf{B}) \\
&= \det(\mathbf{E}_k)\det(\mathbf{E}_{k-1})\cdots\det(\mathbf{E}_1)\det(\mathbf{B}) \\
&= \det(\mathbf{E}_k\mathbf{E}_{k-1}\cdots\mathbf{E}_1)\det(\mathbf{B}) \\
&= \det(\mathbf{A})\det(\mathbf{B}).
\end{aligned}$$

$\square$

Definition

Let $\mathbf{A}$ *be a square matrix. The* **adjoint matrix** *of* $\mathbf{A}$ *is*

$$\mathrm{adj}\mathbf{A} = [A_{ij}]^T,$$

where A_{ij} *is the ij-th cofactor of* $\mathbf{A}$*. In other words,*

$$[\mathrm{adj}\mathbf{A}]_{ij} = A_{ji}.$$

Theorem

Let $\mathbf{A}$ *be a square matrix. Then*

$$\mathbf{A}\mathrm{adj}\mathbf{A} = (\mathrm{adj}\mathbf{A})\mathbf{A} = \det(\mathbf{A})\mathbf{I},$$

where $\mathrm{adj}\mathbf{A}$ *is the adjoint matrix. In particular if* $\mathbf{A}$ *is invertible, then*

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{adj}\mathbf{A}.$$

Proof.

The second statement follows easily from the first. For the first statement, we have

$$\begin{aligned} [\mathbf{A}\mathrm{adj}\mathbf{A}]_{ij} &= \sum_{l=1}^{n} a_{il}[\mathrm{adj}\mathbf{A}]_{lj} \\ &= \sum_{l=1}^{n} a_{il}A_{jl} \\ &= \delta_{ij}\det(\mathbf{A}) \end{aligned}$$

where

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}.$$

Therefore $\mathbf{A}\mathrm{adj}\mathbf{A} = \det(A)\mathbf{I}$ and similarly $(\mathrm{adj}\mathbf{A})\mathbf{A} = \det(A)\mathbf{I}$. $\square$

Example

Let $\mathbf{A} = \begin{pmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{pmatrix}$. *We have*

$$\det(\mathbf{A}) = 4\begin{vmatrix} 6 & 3 \\ 5 & 2 \end{vmatrix} - 3\begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} + 2\begin{vmatrix} 5 & 6 \\ 3 & 5 \end{vmatrix} = 4(-3) - 3(1) + 2(7) = -1,$$

$$\mathrm{adj}\mathbf{A} = \begin{pmatrix} \begin{vmatrix} 6 & 3 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 2 \\ 5 & 2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 6 & 3 \end{vmatrix} \\ -\begin{vmatrix} 5 & 3 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 4 & 2 \\ 5 & 3 \end{vmatrix} \\ \begin{vmatrix} 5 & 6 \\ 3 & 5 \end{vmatrix} & -\begin{vmatrix} 4 & 3 \\ 3 & 5 \end{vmatrix} & \begin{vmatrix} 4 & 3 \\ 5 & 6 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} -3 & 4 & -3 \\ -1 & 2 & -2 \\ 7 & -11 & 9 \end{pmatrix}.$$

Therefore

$$\mathbf{A}^{-1} = \frac{1}{-1}\begin{pmatrix} -3 & 4 & -3 \\ -1 & 2 & -2 \\ 7 & -11 & 9 \end{pmatrix} = \begin{pmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{pmatrix}.$$

Theorem (Cramer's rule)

Consider the $n \times n$ *linear system* $\mathbf{Ax} = \mathbf{b}$, *with*

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}.$$

If $\det(\mathbf{A}) \neq 0$, *then the* i*-th entry of the unique solution* $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ *is*

$$x_i = \det(\mathbf{A})^{-1} \det(\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \end{bmatrix}),$$

where the matrix in the last factor is obtained by replacing the i*-th column of* $\mathbf{A}$ *by* $\mathbf{b}$.

Proof.

$$
\begin{aligned}
x_i &= [\mathbf{A}^{-1}\mathbf{b}]_i \\
&= \frac{1}{\det(\mathbf{A})}[(\mathrm{adj}\mathbf{A})\mathbf{b}]_i \\
&= \frac{1}{\det(\mathbf{A})}\sum_{l=1}^{n} A_{li} b_l \\
&= \frac{1}{\det(\mathbf{A})}\begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}
\end{aligned}
$$

□

Example

Use Cramer's rule to solve the linear system

$$\left\{\begin{array}{rcrcrcl} x_1 & + & 4x_2 & + & 5x_3 & = & 2 \\ 4x_1 & + & 2x_2 & + & 5x_3 & = & 3 \\ -3x_1 & + & 3x_2 & - & x_3 & = & 1 \end{array}\right. .$$

Solution: Let $\mathbf{A} = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{pmatrix}$.

$$\begin{aligned} \det(\mathbf{A}) &= 1\begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} - 4\begin{vmatrix} 4 & 5 \\ -3 & -1 \end{vmatrix} + 5\begin{vmatrix} 4 & 2 \\ -3 & 3 \end{vmatrix} \\ &= 1(-17) - 4(11) + 5(18) \\ &= 29. \end{aligned}$$

Thus by Cramer's rule,

$$\begin{aligned}
x_1 &= \frac{1}{29}\begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 5 \\ 1 & 3 & -1 \end{vmatrix} = \frac{33}{29} \\
x_2 &= \frac{1}{29}\begin{vmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \\ -3 & 1 & -1 \end{vmatrix} = \frac{35}{29} \\
x_3 &= \frac{1}{29}\begin{vmatrix} 1 & 4 & 2 \\ 4 & 2 & 3 \\ -3 & 3 & 1 \end{vmatrix} = -\frac{23}{29}
\end{aligned}$$

$\square$

Theorem

Let n be a non-negative integer, and $(x_0, y_0), (x_1, y_1), \cdots, (x_n, y_n)$ be $n+1$ points in $\mathbb{R}^2$ such that $x_i \neq x_j$ for any $i \neq j$. Then there exists unique polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

of degree at most n such that $p(x_i) = y_i$ for all $0 \leq i \leq n$. The coefficients of $p(x)$ satisfy the linear system

$$\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Theorem

Moreover, we can write down the polynomial function $y = p(x)$ directly as

$$\begin{vmatrix} 1 & x & x^2 & \cdots & x^n & y \\ 1 & x_0 & x_0^2 & \cdots & x_0^n & y_0 \\ 1 & x_1 & x_1^2 & \cdots & x_1^n & y_1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^n & y_n \end{vmatrix} = 0.$$

Proof.

Expanding the determinant, one sees that the equation is of the form $y = p(x)$ where $p(x)$ is a polynomial of degree at most n. Observe that the determinant is zero when $(x, y) = (x_i, y_i)$ for some $0 \leq i \leq n$ since two rows would be identical in this case. Now it is well known that such polynomial is unique. $\square$

Example

Find the equation of straight line passes through the points (x_0, y_0) and (x_1, y_1).

Solution: The equation of the required straight line is

$$\begin{aligned}\begin{vmatrix} 1 & x & y \\ 1 & x_0 & y_0 \\ 1 & x_1 & y_1 \end{vmatrix} &= 0 \\ (y_0 - y_1)x + (x_1 - x_0)y + (x_0y_1 - x_1y_0) &= 0\end{aligned}$$

$\square$

Example

Find the cubic polynomial that interpolates the data points $(-1,4),(1,2),(2,1)$ *and* $(3,16)$.

Solution: The required equation is

$$\begin{vmatrix} 1 & x & x^2 & x^3 & y \\ 1 & -1 & 1 & -1 & 4 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 1 \\ 1 & 3 & 9 & 27 & 16 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & x^2 & x^3 & y \\ 1 & -1 & 1 & -1 & 4 \\ 0 & 2 & 0 & 2 & -2 \\ 0 & 3 & 3 & 9 & -3 \\ 0 & 4 & 8 & 28 & 12 \end{vmatrix} = 0$$

$$\vdots$$

$$\begin{vmatrix} 1 & x & x^2 & x^3 & y \\ 1 & 0 & 0 & 0 & 7 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -4 \\ 0 & 0 & 0 & 1 & 2 \end{vmatrix} = 0$$

$$-7 + 3x + 4x^2 - 2x^3 + y = 0$$

$$y = 7 - 3x - 4x^2 + 2x^3$$

Example

Find the equation of the circle determined by the points $(-1,5),(5,-3)$ *and* $(6,4)$.

Solution: The equation of the required circle is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ (-1)^2+5^2 & -1 & 5 & 1 \\ 5^2+(-3)^2 & 5 & -3 & 1 \\ 6^2+4^2 & 6 & 4 & 1 \end{vmatrix} = 0$$

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 26 & -1 & 5 & 1 \\ 34 & 5 & -3 & 1 \\ 52 & 6 & 4 & 1 \end{vmatrix} = 0$$

$$\vdots$$

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ 20 & 0 & 0 & 1 \\ 4 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \end{vmatrix} = 0$$

$$x^2+y^2-4x-2y-20 = 0$$