THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5520
Differential Equations & Linear Algebra
Suggested Solution for Assignment 4
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Exercise 5.2 Question 1(b)

Diagonalize the following matrices.

$$\begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix}$$

Solution: Denote the given matrix by $\mathbf{A}$. Solving the characteristic equation, we have

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= 0 \\ \begin{vmatrix} \lambda - 3 & 2 \\ -4 & \lambda + 1 \end{vmatrix} &= 0 \\ \lambda^2 - 2\lambda + 5 &= 0 \\ \lambda &= 1 \pm 2i. \end{aligned}$$

For $\lambda_1 = 1 + 2i$,

$$\begin{aligned} ((1+2i)\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -2+2i & 2 \\ -4 & 2+2i \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_1 = (1, 1-i)^T$ is an eigenvector associated with $\lambda_1 = 1 + 2i$.
For $\lambda_2 = 1 - 2i$,

$$\begin{aligned} ((1-2i)\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -2+2i & 2 \\ -4 & 2+2i \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_2 = (1, 1+i)^T$ is an eigenvector associated with $\lambda_2 = 1 - 2i$.
Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1-i & 1+i \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 1 & 1 \\ 1-i & 1+i \end{pmatrix}^{-1} \begin{pmatrix} 3 & -2 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1-i & 1+i \end{pmatrix} \\ &= \begin{pmatrix} 1+2i & 0 \\ 0 & 1-2i \end{pmatrix}. \end{aligned}$$

□

Exercise 5.2 Question 1(d)

Diagonalize the following matrices.

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 6 & 11 & 6 \end{pmatrix}$$

Solution: Denote the given matrix by $\mathbf{A}$. Solving the characteristic equation, we have

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= 0 \\ \begin{vmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ -6 & -11 & \lambda - 6 \end{vmatrix} &= 0 \\ \lambda^3 - 6\lambda^2 + 11\lambda - 6 &= 0 \\ \lambda &= 1, 2, 3. \end{aligned}$$

For $\lambda_1 = 1$,

$$\begin{aligned} (\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -6 & -11 & -5 \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_1 = (1, -1, 1)^T$ is an eigenvector associated with $\lambda_1 = 1$.

For $\lambda_2 = 2$,

$$\begin{aligned} (2\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ -6 & -11 & -4 \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_2 = (1, -2, 4)^T$ is an eigenvector associated with $\lambda_2 = 2$.

For $\lambda_3 = 3$,

$$\begin{aligned} (3\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -6 & -11 & -3 \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_3 = (1, -3, 9)^T$ is an eigenvector associated with $\lambda_3 = 3$.

Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\begin{aligned}\mathbf{P}^{-1}\mathbf{AP} &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 6 & 11 & 6 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.\end{aligned}$$

□

Exercise 5.2 Question 1(e)

Diagonalize the following matrices.

$$\begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{pmatrix}$$

Solution: Denote the given matrix by $\mathbf{A}$. Solving the characteristic equation, we have

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= 0 \\ \begin{vmatrix} \lambda - 3 & 2 & 0 \\ 0 & \lambda - 1 & 0 \\ 4 & -4 & \lambda - 1 \end{vmatrix} &= 0 \\ \lambda^3 - 5\lambda^2 + 7\lambda - 3 &= 0 \\ \lambda &= 1, 1, 3. \end{aligned}$$

For $\lambda_1 = \lambda_2 = 1$,

$$\begin{aligned} (\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 4 & -4 & 0 \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_1 = (0, 0, 1)^T$ and $\mathbf{v}_2 = (1, 1, 0)^T$ are two eigenvectors associated with $\lambda_1 = \lambda_2 = 1$. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ constitutes a basis for the eigenspace associated with eigenvalue $\lambda_1 = \lambda_2 = 1$.

For $\lambda_3 = 3$,

$$\begin{aligned} (3\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 4 & -4 & 2 \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_3 = (-1, 0, 2)^T$ is an eigenvector associated with $\lambda_3 = 3$.

Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ -4 & 4 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \end{aligned}$$

□

Exercise 5.2 Question 2(a)

Show that that following matrices are not diagonalizable.

$$\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

Solution: Denote the given matrix by $\mathbf{A}$. Solving the characteristic equation, we have

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= 0 \\ \begin{vmatrix} \lambda - 3 & -1 \\ 1 & \lambda - 1 \end{vmatrix} &= 0 \\ \lambda^2 - 4\lambda + 4 &= 0 \\ \lambda &= 2, 2. \end{aligned}$$

For $\lambda_1 = \lambda_2 = 2$,

$$\begin{aligned} (2\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_1 = (-1, 1)^T$ is an eigenvector associated with $\lambda_1 = \lambda_2 = 2$. Thus $\{\mathbf{v}_1\}$ constitutes a basis for the eigenspace associated with eigenvalue $\lambda_1 = \lambda_2 = 2$. Note that $\lambda_1 = \lambda_2 = 2$ is a root of multiplicity two of the characteristic equation, but we can only find one linearly independent eigenvector for $\lambda_1 = \lambda_2 = 2$. Therefore $\mathbf{A}$ is not diagonalizable. □

Exercise 5.2 Question 2(b)

Show that that following matrices are not diagonalizable.

$$\begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution: Denote the given matrix by $\mathbf{A}$. Solving the characteristic equation, we have

$$\det(\lambda\mathbf{I} - \mathbf{A}) = 0$$
$$\begin{vmatrix} \lambda+1 & -1 & 0 \\ 4 & \lambda-3 & 0 \\ -1 & 0 & \lambda-2 \end{vmatrix} = 0$$
$$\lambda^3 - 4\lambda^2 + 5\lambda - 2 = 0$$
$$\lambda = 1, 1, 2.$$

For $\lambda_1 = \lambda_2 = 1$,

$$(\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$$
$$\begin{pmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ -1 & 0 & -1 \end{pmatrix}\mathbf{v} = \mathbf{0}$$

Thus $\mathbf{v}_1 = (-1, -2, 1)^T$ is an eigenvector associated with $\lambda_1 = \lambda_2 = 1$. Thus $\{\mathbf{v}_1\}$ constitutes a basis for the eigenspace associated with eigenvalue $\lambda_1 = \lambda_2 = 1$. Note that $\lambda_1 = \lambda_2 = 1$ is a root of multiplicity two of the characteristic equation, but we can only find one linearly independent eigenvector for $\lambda_1 = \lambda_2 = 1$. Therefore $\mathbf{A}$ is not diagonalizable. □

Exercise 5.2 Question 7

Let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a 2×2 matrix. Show that if $(a-d)^2 + 4bc \neq 0$, then $\mathbf{A}$ is diagonalizable.

Solution: The characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{aligned} p(x) &= \det(x\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} x-a & -b \\ -c & x-d \end{vmatrix} \\ &= (x-a)(x-d) - (-b)(-c) \\ &= x^2 - (a+d)x + (ad-bc). \end{aligned}$$

The discriminant of $p(x)$ for the characteristic equation $p(x) = 0$ is given by

$$\begin{aligned} \Delta &= (-(a+d))^2 - 4(1)(ad-bc) \\ &= a^2 + 2ad + d^2 - 4ad + 4bc \\ &= (a-d)^2 + 4bc. \end{aligned}$$

Suppose $(a-d)^2 + 4bc \neq 0$. Then $\Delta \neq 0$ and $p(x) = 0$ has 2 distinct roots. This implies the 2×2 matrix $\mathbf{A}$ has 2 distinct eigenvalues. Hence, by Theorem 5.2.12, $\mathbf{A}$ is diagonalizable. □

Exercise 5.2 Question 10

Prove that if $\mathbf{A}$ is a non-singular matrix, then for any matrix $\mathbf{B}$, we have $\mathbf{AB}$ is similar to $\mathbf{BA}$.

Solution: Take $\mathbf{P} = \mathbf{A}$. Then $\mathbf{P}$ is non-singular and

$$\mathbf{P}^{-1}(\mathbf{AB})\mathbf{P} = \mathbf{A}^{-1}(\mathbf{AB})\mathbf{A} = (\mathbf{A}^{-1}\mathbf{A})(\mathbf{BA}) = \mathbf{I}(\mathbf{BA}) = \mathbf{BA}.$$

Hence $\mathbf{AB}$ is similar to $\mathbf{BA}$. □

Exercise 5.3 Question 1(a)

Compute $\mathbf{A}^5$ where $\mathbf{A}$ is the given matrix.

$$\begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$$

Solution: Solving the characteristic equation, we have

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= 0 \\ \begin{vmatrix} \lambda - 5 & 6 \\ -3 & \lambda + 4 \end{vmatrix} &= 0 \\ \lambda^2 - \lambda + 2 &= 0 \\ \lambda &= -1, 2. \end{aligned}$$

For $\lambda_1 = -1$,

$$\begin{aligned} (-\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -6 & 6 \\ 3 & -3 \end{pmatrix}\mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_1 = (1,1)^T$ is an eigenvector associated with $\lambda_1 = -1$.

For $\lambda_2 = 2$,

$$\begin{aligned} (2\mathbf{I} - \mathbf{A})\mathbf{v} &= \mathbf{0} \\ \begin{pmatrix} -3 & 6 \\ -3 & 6 \end{pmatrix}\mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_2 = (2,1)^T$ is an eigenvector associated with $\lambda_2 = 2$.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{AP} &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} = \mathbf{D}. \end{aligned}$$

Hence

$$\mathbf{A} = \mathbf{PDP}^{-1}$$

and therefore

$$\begin{aligned} \mathbf{A}^5 &= (\mathbf{PDP}^{-1})^5 \\ &= \mathbf{PD}^5\mathbf{P}^{-1} \\ &= \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 32 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 65 & -66 \\ 33 & -34 \end{pmatrix}. \end{aligned}$$

□

Exercise 5.3 Question 1(e)

Compute $\mathbf{A}^5$ where $\mathbf{A}$ is the given matrix.

$$\begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix}$$

Solution: Solving the characteristic equation, we have

$$\begin{aligned} \det(\lambda\mathbf{I} - \mathbf{A}) &= 0 \\ \begin{vmatrix} \lambda - 1 & -2 & 1 \\ -2 & \lambda - 4 & 2 \\ -3 & -6 & \lambda + 3 \end{vmatrix} &= 0 \\ \lambda^3 - 2\lambda^2 &= 0 \\ \lambda &= 0, 0, 2. \end{aligned}$$

For $\lambda_1 = \lambda_2 = 0$,

$$\begin{aligned} -\mathbf{Av} &= \mathbf{0} \\ \begin{pmatrix} -1 & -2 & 1 \\ -2 & -4 & 2 \\ -3 & -6 & 3 \end{pmatrix} \mathbf{v} &= \mathbf{0} \end{aligned}$$

Thus $\mathbf{v}_1 = (-2,1,0)^T$ and $\mathbf{v}_2 = (1,0,1)^T$ are two eigenvectors associated with $\lambda_1 = \lambda_2 = 0$. The set $\{\mathbf{v}_1, \mathbf{v}_2\}$ constitutes a basis for the eigenspace associated with eigenvalue $\lambda_1 = \lambda_2 = 0$.

For $\lambda_3 = 2$,

$$(2\mathbf{I} - \mathbf{A})\mathbf{v} = \mathbf{0}$$
$$\begin{pmatrix} 1 & -2 & 1 \\ -2 & -2 & 2 \\ -3 & -6 & 5 \end{pmatrix} \mathbf{v} = \mathbf{0}$$

Thus $\mathbf{v}_3 = (1,2,3)^T$ is an eigenvector associated with $\lambda_3 = 2$.

Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent, the matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

diagonalizes $\mathbf{A}$ and

$$\begin{aligned} \mathbf{P}^{-1}\mathbf{A}\mathbf{P} &= \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \\ 3 & 6 & -3 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \mathbf{D}. \end{aligned}$$

Hence

$$\mathbf{A} = \mathbf{P}\mathbf{D}\mathbf{P}^{-1}$$

and therefore

$$\begin{aligned} \mathbf{A}^5 &= (\mathbf{P}\mathbf{D}\mathbf{P}^{-1})^5 \\ &= \mathbf{P}\mathbf{D}^5\mathbf{P}^{-1} \\ &= \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} -2 & 1 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 16 & 32 & -16 \\ 32 & 64 & -32 \\ 48 & 96 & -48 \end{pmatrix}. \end{aligned}$$

□

Exercise 5.4 Question 1(a)

Find the minimal polynomial of $\mathbf{A}$ where $\mathbf{A}$ is the matrix given below. Then express $\mathbf{A}^4$ and $\mathbf{A}^{-1}$ as a polynomial in $\mathbf{A}$ of smallest degree.

$$\begin{pmatrix} 5 & -4 \\ 3 & -2 \end{pmatrix}$$

Solution: The characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{aligned} p(x) &= \det(x\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} x-5 & 4 \\ -3 & x+2 \end{vmatrix} \\ &= (x-1)(x-2). \end{aligned}$$

The minimal polynomial of $\mathbf{A}$ is of the form

$$m(x) = (x+1)^{m_1}(x-2)^{m_2},$$

where $m_1 = 1$ and $m_2 = 1$. Therefore the minimal polynomial of $\mathbf{A}$ is its characteristic polynomial

$$m(x) = (x-1)(x-2) = p(x).$$

We have

$$\begin{aligned} & \mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I}, \\ \Longrightarrow\ & \mathbf{A}^4 = (\mathbf{A}^2)^2 \\ &\quad = (3\mathbf{A} - 2\mathbf{I})^2 \\ &\quad = 9\mathbf{A}^2 - 12\mathbf{A} + 4\mathbf{I} \\ &\quad = 9(3\mathbf{A} - 2\mathbf{I}) - 12\mathbf{A} + 4\mathbf{I} \\ &\quad = 15\mathbf{A} - 14\mathbf{I}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbf{A}^2 = 3\mathbf{A} - 2\mathbf{I}, \\ \Longrightarrow\ & \mathbf{A} = 3\mathbf{I} - 2\mathbf{A}^{-1}, \\ \Longrightarrow\ & \mathbf{A}^{-1} = -\frac{1}{2}\mathbf{A} + \frac{3}{2}\mathbf{I}. \end{aligned}$$

□

Exercise 5.4 Question 1(b)

Find the minimal polynomial of $\mathbf{A}$ where $\mathbf{A}$ is the matrix given below. Then express $\mathbf{A}^4$ and $\mathbf{A}^{-1}$ as a polynomial in $\mathbf{A}$ of smallest degree.

$$\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

Solution: The characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{aligned} p(x) &= \det(x\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} x-3 & 2 \\ -2 & x+1 \end{vmatrix} \\ &= (x-1)^2. \end{aligned}$$

The minimal polynomial of $\mathbf{A}$ is of the form

$$m(x) = (x-1)^{m_1},$$

where $1 \leq m_1 \leq 2$. Therefore the minimal polynomial of $\mathbf{A}$ is either

$$m(x) = x - 1 \quad \text{or} \quad m(x) = (x-1)^2 = p(x).$$

By direct computation,

$$\mathbf{A} - \mathbf{I} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{0}.$$

Hence the minimal polynomial of $\mathbf{A}$ is

$$m(x) = (x-1)^2.$$

We have

$$\begin{aligned} \mathbf{A}^2 &= 2\mathbf{A} - \mathbf{I}, \\ \Longrightarrow \mathbf{A}^4 &= (\mathbf{A}^2)^2 \\ &= (2\mathbf{A} - \mathbf{I})^2 \\ &= 4\mathbf{A}^2 - 4\mathbf{A} + \mathbf{I} \\ &= 4(2\mathbf{A} - \mathbf{I}) - 4\mathbf{A} + \mathbf{I} \\ &= 4\mathbf{A} - 3\mathbf{I}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \mathbf{A}^2 &= 2\mathbf{A} - \mathbf{I}, \\ \Longrightarrow \mathbf{A} &= 2\mathbf{I} - \mathbf{A}^{-1}, \\ \Longrightarrow \mathbf{A}^{-1} &= -\mathbf{A} + 2\mathbf{I}. \end{aligned}$$

□

Exercise 5.4 Question 1(d)

Find the minimal polynomial of $\mathbf{A}$ where $\mathbf{A}$ is the matrix given below. Then express $\mathbf{A}^4$ and $\mathbf{A}^{-1}$ as a polynomial in $\mathbf{A}$ of smallest degree.

$$\begin{pmatrix} -1 & 1 & 0 \\ -4 & 3 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Solution: The characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{aligned} p(x) &= \det(x\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} x+1 & -1 & 0 \\ 4 & x-3 & 0 \\ -1 & 0 & x-2 \end{vmatrix} \\ &= (x-1)^2(x-2). \end{aligned}$$

The minimal polynomial of $\mathbf{A}$ is of the form

$$m(x) = (x-1)^{m_1}(x-2)^{m_2},$$

where $1 \leq m_1 \leq 2$ and $m_2 = 1$. Therefore the minimal polynomial of $\mathbf{A}$ is either

$$m(x) = (x-1)(x-2) \quad \text{or} \quad m(x) = (x-1)^2(x-2) = p(x).$$

By direct computation,

$$\begin{aligned} (\mathbf{A} - \mathbf{I})(\mathbf{A} - 2\mathbf{I}) &= \begin{pmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -3 & 1 & 0 \\ -4 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -1 & 0 \\ 4 & -2 & 0 \\ -2 & 1 & 0 \end{pmatrix} \\ &\neq \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}. \end{aligned}$$

Hence the minimal polynomial of $\mathbf{A}$ is

$$m(x) = (x-1)^2(x-2).$$

We have

$$\begin{aligned} & \mathbf{A}^3 = 4\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I}, \\ \Longrightarrow \; & \mathbf{A}^4 = (\mathbf{A})(\mathbf{A}^3) \\ & \quad = (\mathbf{A})(4\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I}) \\ & \quad = 4\mathbf{A}^3 - 5\mathbf{A}^2 + 2\mathbf{A} \\ & \quad = 4(4\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I}) - 5\mathbf{A}^2 + 2\mathbf{A} \\ & \quad = 11\mathbf{A}^2 - 18\mathbf{A} + 8\mathbf{I}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbf{A}^3 = 4\mathbf{A}^2 - 5\mathbf{A} + 2\mathbf{I}, \\ \Longrightarrow \; & \mathbf{A}^2 = 4\mathbf{A} - 5\mathbf{I} + 2\mathbf{A}^{-1}, \\ \Longrightarrow \; & \mathbf{A}^{-1} = \frac{1}{2}\mathbf{A}^2 - 2\mathbf{A} + \frac{5}{2}\mathbf{I}. \end{aligned}$$

□

Exercise 5.4 Question 1(e)

Find the minimal polynomial of $\mathbf{A}$ where $\mathbf{A}$ is the matrix given below. Then express $\mathbf{A}^4$ and $\mathbf{A}^{-1}$ as a polynomial in $\mathbf{A}$ of smallest degree.

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

Solution: The characteristic polynomial of $\mathbf{A}$ is given by

$$\begin{aligned} p(x) &= \det(x\mathbf{I} - \mathbf{A}) \\ &= \begin{vmatrix} x-3 & -1 & -1 \\ -2 & x-4 & -2 \\ 1 & 1 & x-1 \end{vmatrix} \\ &= (x-2)^2(x-4). \end{aligned}$$

The minimal polynomial of $\mathbf{A}$ is of the form

$$m(x) = (x-2)^{m_1}(x-4)^{m_2},$$

where $1 \leq m_1 \leq 2$ and $m_2 = 1$. Therefore the minimal polynomial of $\mathbf{A}$ is either

$$m(x) = (x-2)(x-4) \qquad \text{or} \qquad m(x) = (x-2)^2(x-4) = p(x).$$

By direct computation,

$$(\mathbf{A} - 2\mathbf{I})(\mathbf{A} - 4\mathbf{I}) = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \mathbf{0}.$$

Hence the minimal polynomial of $\mathbf{A}$ is

$$m(x) = (x-2)(x-4).$$

We have

$$\begin{aligned} & \mathbf{A}^2 = 6\mathbf{A} - 8\mathbf{I}, \\ \Longrightarrow\ & \mathbf{A}^4 = (\mathbf{A}^2)^2 \\ &\quad = (6\mathbf{A} - 8\mathbf{I})^2 \\ &\quad = 36\mathbf{A}^2 - 96\mathbf{A} + 64\mathbf{I} \\ &\quad = 36(6\mathbf{A} - 8\mathbf{I}) - 96\mathbf{A} + 64\mathbf{I} \\ &\quad = 120\mathbf{A} - 224\mathbf{I}. \end{aligned}$$

On the other hand,

$$\begin{aligned} & \mathbf{A}^2 = 6\mathbf{A} - 8\mathbf{I}, \\ \Longrightarrow\ & \mathbf{A} = 6\mathbf{I} - 8\mathbf{A}^{-1}, \\ \Longrightarrow\ & \mathbf{A}^{-1} = -\frac{1}{8}\mathbf{A} + \frac{3}{4}\mathbf{I}. \end{aligned}$$

□