THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT5520
Differential Equations & Linear Algebra
Suggested Solution for Assignment 2
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Exercise 2.1 Question 2(c)

Solve the following systems of linear equations.

$$\begin{cases} 2x_1 & - & x_2 & + & 5x_3 & = & 15 \\ x_1 & + & 3x_2 & - & x_3 & = & 4 \\ x_1 & - & 4x_2 & + & 6x_3 & = & 11 \\ 3x_1 & + & 9x_2 & - & 3x_3 & = & 12 \end{cases}$$

Solution:

$$\begin{aligned}
&\left(\begin{array}{ccc|c} 2 & -1 & 5 & 15 \\ 1 & 3 & -1 & 4 \\ 1 & -4 & 6 & 11 \\ 3 & 9 & -3 & 12 \end{array}\right) \\
\xrightarrow{R_1 \to \frac{1}{2}R_1} &\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{2} & \frac{15}{2} \\ 1 & 3 & -1 & 4 \\ 1 & -4 & 6 & 11 \\ 3 & 9 & -3 & 12 \end{array}\right) \\
\xrightarrow[R_4 \to R_4 - R_1]{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}} &\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{2} & \frac{15}{2} \\ 0 & \frac{7}{2} & -\frac{7}{2} & -\frac{7}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 0 & \frac{21}{2} & -\frac{21}{2} & -\frac{21}{2} \end{array}\right) \\
\xrightarrow{R_2 \to \frac{2}{7}R_2} &\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{5}{2} & \frac{15}{2} \\ 0 & 1 & -1 & -1 \\ 0 & -\frac{7}{2} & \frac{7}{2} & \frac{7}{2} \\ 0 & \frac{21}{2} & -\frac{21}{2} & -\frac{21}{2} \end{array}\right) \\
\xrightarrow[R_4 \to R_4 - \frac{21}{2}R_2]{\substack{R_1 \to R_1 + R_2 \\ R_3 \to R_3 + \frac{7}{2}R_2}} &\left(\begin{array}{ccc|c} 1 & 0 & 2 & 7 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}$$

Now x_1 and x_2 are leading variables, while x_3 is a free variable. The solution of the system is

$$(x_1, x_2, x_3) = (7 - 2\alpha, -1 + \alpha, \alpha), \text{ where } \alpha \in \mathbb{R}.$$

$\square$

Exercise 2.1 Question 2(e)

Solve the following systems of linear equations.

$$\begin{cases} x_1 - 2x_2 + x_3 + x_4 = 1 \\ x_1 - 2x_2 + x_3 - x_4 = -1 \\ x_1 - 2x_2 + x_3 + 5x_4 = 5 \end{cases}$$

Solution:

$$\begin{aligned}
&\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & -1 & -1 \\ 1 & -2 & 1 & 5 & 5 \end{array}\right) \\
\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} &\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right) \\
\xrightarrow{R_2 \to -\frac{1}{2}R_2} &\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 & 4 \end{array}\right) \\
\xrightarrow[R_3 \to R_3 - 4R_2]{R_1 \to R_1 - R_2} &\left(\begin{array}{cccc|c} 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right).
\end{aligned}$$

Now x_1 and x_4 are leading variables, while x_2 and x_3 are free variables. The solution of the system is

$$(x_1, x_2, x_3, x_4) = (2\alpha - \beta, \alpha, \beta, 1), \text{ where } \alpha, \beta \in \mathbb{R}.$$

□

Exercise 2.4 Question 3(b)

For the given matrix $\mathbf{A}$, evaluate $\mathbf{A}^{-1}$ by finding the adjoint matrix adj$\mathbf{A}$ of $\mathbf{A}$.

$$\mathbf{A} = \begin{pmatrix} 2 & -3 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

Solution: Since $\mathbf{A}$ is a triangular matrix, the determinant of $\mathbf{A}$ is the product of the diagonal elements of $\mathbf{A}$, i.e.

$$\det(\mathbf{A}) = \prod_{i=1}^{3} a_{ii} = (2)(1)(2) = 4.$$

The adjoint matrix of $\mathbf{A}$ is given by

$$\mathrm{adj}\mathbf{A} = \begin{pmatrix} \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 5 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} -3 & 5 \\ 1 & -3 \end{vmatrix} \\ -\begin{vmatrix} 0 & -3 \\ 0 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 0 & -3 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ 0 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -3 \\ 0 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix}.$$

Therefore

$$\mathbf{A}^{-1} = \frac{1}{\det(\mathbf{A})}\mathrm{adj}\mathbf{A} = \frac{1}{4}\begin{pmatrix} 2 & 6 & 4 \\ 0 & 4 & 6 \\ 0 & 0 & 2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}.$$

$\square$

Exercise 2.4 Question 4(a)

Use Cramers Rule to solve the following linear systems.

$$\begin{cases} 4x_1 & - & x_2 & - & x_3 & = & 1 \\ 2x_1 & + & 2x_2 & + & 3x_3 & = & 10 \\ 5x_1 & - & 2x_2 & - & 2x_3 & = & -1 \end{cases}$$

Solution: The determinant of $\mathbf{A}$ is given by

$$\det(\mathbf{A}) = \begin{vmatrix} 4 & -1 & -1 \\ 2 & 2 & 3 \\ 5 & -2 & -2 \end{vmatrix} = 3.$$

By Cramer's rule, we have

$$x_1 = \frac{1}{3} \begin{vmatrix} 1 & -1 & -1 \\ 10 & 2 & 3 \\ -1 & -2 & -2 \end{vmatrix} = 1,$$

$$x_2 = \frac{1}{3} \begin{vmatrix} 4 & 1 & -1 \\ 2 & 10 & 3 \\ 5 & -1 & -2 \end{vmatrix} = 1,$$

$$x_3 = \frac{1}{3} \begin{vmatrix} 4 & -1 & 1 \\ 2 & 2 & 10 \\ 5 & -2 & -1 \end{vmatrix} = 2.$$

$\square$

Exercise 2.5 Question 1(a)

Find the equation of the parabola of the form $y = ax^2 + bx + c$ passing through the given set of three points.

$$(0, -5), (2, -1), (3, 4)$$

Solution: The required equation is

$$\begin{vmatrix} 1 & x & x^2 & y \\ 1 & 0 & 0^2 & -5 \\ 1 & 2 & 2^2 & -1 \\ 1 & 3 & 3^2 & 4 \end{vmatrix} = 0$$

$$\begin{vmatrix} 1 & x & x^2 & y \\ 1 & 0 & 0 & -5 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 4 \end{vmatrix} = 0.$$

Expanding the determinant along the second row, we have

$$-\left((1) \begin{vmatrix} x & x^2 & y \\ 2 & 4 & -1 \\ 3 & 9 & 4 \end{vmatrix} - (-5) \begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} \right) = 0$$

$$\left(x \begin{vmatrix} 4 & -1 \\ 9 & 4 \end{vmatrix} - x^2 \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix} + y \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} \right) + 5(2-x)(3-x)(3-2) = 0$$

$$(25x - 11x^2 + 6y) + 5(6 - 5x + x^2) = 0$$

$$6y - 6x^2 + 30 = 0$$

$$y = x^2 - 5.$$

Remark:

- We used the so-called Laplace expansion which allows us to expand the determinant along not only the first row but any other row. (See Theorem 2.6 in Lecture Notes). We choose the second row because there are the most number of zeros.
- We quickly obtain

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = (2-x)(3-x)(3-2)$$

  since it is in the form of a Vandermonde determinant. (See Example 2.4.12 of Lecture Notes).

$\square$

Exercise 2.5 Question 2(a)

Find the equation of the circle passing through the given set of three points.

$$(-1,-1),(6,6),(7,5)$$

Solution: The required equation is

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & -1 & -1 & (-1)^2+(-1)^2 \\ 1 & 6 & 6 & 6^2+6^2 \\ 1 & 7 & 5 & 7^2+5^2 \end{vmatrix} = 0$$
$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & -1 & -1 & 2 \\ 1 & 6 & 6 & 72 \\ 1 & 7 & 5 & 74 \end{vmatrix} = 0.$$

Note that elementary row operations preserve singular matrices, i.e. if $\mathbf{A}$ is singular and $\mathbf{E}$ is an elementary matrix, then $\det(\mathbf{EA}) = \det(\mathbf{E})\det(\mathbf{A}) = 0$ and $\mathbf{EA}$ is still singular. Using elementary row operations, the equation is reduced to

$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & -1 & -1 & 2 \\ 1 & 6 & 6 & 72 \\ 1 & 7 & 5 & 74 \end{vmatrix} = 0$$
$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & -1 & -1 & 2 \\ 0 & 7 & 7 & 70 \\ 0 & 8 & 6 & 72 \end{vmatrix} = 0$$
$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & -1 & -1 & 2 \\ 0 & 1 & 1 & 10 \\ 0 & 8 & 6 & 72 \end{vmatrix} = 0$$
$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & -2 & -8 \end{vmatrix} = 0$$
$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 1 & 10 \\ 0 & 0 & 1 & 4 \end{vmatrix} = 0$$
$$\begin{vmatrix} 1 & x & y & x^2+y^2 \\ 1 & 0 & 0 & 12 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 4 \end{vmatrix} = 0.$$

Expanding the determinant along the second row, we have

$$
\begin{aligned}
-\left((1)\begin{vmatrix} x & y & x^2+y^2 \\ 1 & 0 & 6 \\ 0 & 1 & 4 \end{vmatrix} - (12)\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}\right) &= 0 \\
-\left(x\begin{vmatrix} 0 & 6 \\ 1 & 4 \end{vmatrix} - y\begin{vmatrix} 1 & 6 \\ 0 & 4 \end{vmatrix} + (x^2+y^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right) + 12(1) &= 0 \\
-(-6x-4y+(x^2+y^2))+12 &= 0 \\
-x^2-y^2+6x+4y+12 &= 0 \\
x^2+y^2-6x-4y-12 &= 0.
\end{aligned}
$$

□

Exercise 3.3 Question 1(e)

Determine whether the given set of vectors are linearly independent in $\mathbb{R}^3$.

$$\mathbf{v}_1 = (3, -1, -2), \mathbf{v}_2 = (2, 0, -1), \mathbf{v}_3 = (1, -3, -2).$$

Solution: Writing the equation $c_1\mathbf{v}_1^T + c_2\mathbf{v}_2^T + c_3\mathbf{v}_3^T = \mathbf{0}^T$ as a system

$$\left\{\begin{array}{rcrcrl} 3c_1 & + & 2c_2 & + & c_3 & = 0 \\ -c_1 & & & - & 3c_3 & = 0 \\ -2c_1 & - & c_2 & - & 2c_3 & = 0 \end{array}\right. .$$

The augmented matrix of the system

$$\left[\begin{array}{rrr|r} 3 & 2 & 1 & 0 \\ -1 & 0 & -3 & 0 \\ -2 & -1 & -2 & 0 \end{array}\right]$$

reduces to the reduced row echelon form

$$\left[\begin{array}{rrr|r} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Thus the system has a nontrivial solution

$$(c_1, c_2, c_3) = (-3, 4, 1).$$

Therefore the set of vectors is linearly dependent.

Alternative solution: Write the vectors into a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1^T & \mathbf{v}_2^T & \mathbf{v}_3^T \end{bmatrix} = \begin{bmatrix} 3 & 2 & 1 \\ -1 & 0 & -3 \\ -2 & -1 & -2 \end{bmatrix}.$$

Expanding along the second row, the determinant of $\mathbf{A}$ is given by

$$\begin{aligned} \det(\mathbf{A}) &= -\left((-1)\begin{vmatrix} 2 & 1 \\ -1 & -2 \end{vmatrix} + (-3)\begin{vmatrix} 3 & 2 \\ -2 & -1 \end{vmatrix}\right) \\ &= -((-1)(-3) + (-3)(1)) \\ &= 0. \end{aligned}$$

Hence the set of vectors is linearly dependent. $\square$

Exercise 3.3 Question 1(f)

Determine whether the given set of vectors are linearly independent in $\mathbb{R}^3$.

$$\mathbf{v}_1 = (1, -2, 2), \mathbf{v}_2 = (3, 0, 1), \mathbf{v}_3 = (1, -1, 2).$$

Solution: Writing the equation $c_1\mathbf{v}_1^T + c_2\mathbf{v}_2^T + c_3\mathbf{v}_3^T = \mathbf{0}^T$ as a system

$$\begin{cases} c_1 & + & 3c_2 & + & c_3 & = 0 \\ -2c_1 & & & - & c_3 & = 0 \\ 2c_1 & + & c_2 & + & 2c_3 & = 0 \end{cases}.$$

The augmented matrix of the system

$$\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ -2 & 0 & -1 & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

reduces to the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

Thus the system has only the trivial solution

$$(c_1, c_2, c_3) = (0, 0, 0).$$

Therefore the set of vectors is linearly independent.

Alternative solution: Write the vectors into a 3×3 matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{v}_1^T & \mathbf{v}_2^T & \mathbf{v}_3^T \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Expanding along the second row, the determinant of $\mathbf{A}$ is given by

$$\begin{aligned} \det(\mathbf{A}) &= -\left((-2)\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\right) \\ &= -((-2)(5) + (-1)(-5)) \\ &= 5 \neq 0. \end{aligned}$$

Hence the set of vectors is linearly independent. $\square$

Exercise 3.5 Question 1(c)

Find a basis for the null space, a basis for the row space and a basis for the column space for the given matrices.

$$\begin{pmatrix} 3 & -6 & 1 & 3 & 4 \\ 1 & -2 & 0 & 1 & 2 \\ 1 & -2 & 2 & 0 & 3 \end{pmatrix}$$

Solution: By Gaussian elimination, we have

$$\begin{aligned}
\mathbf{A} = & \begin{pmatrix} 3 & -6 & 1 & 3 & 4 \\ 1 & -2 & 0 & 1 & 2 \\ 1 & -2 & 2 & 0 & 3 \end{pmatrix} \\
\xrightarrow{R_1 \to \frac{1}{3}R_1} & \begin{pmatrix} 1 & -2 & \frac{1}{3} & 1 & \frac{4}{3} \\ 1 & -2 & 0 & 1 & 2 \\ 1 & -2 & 2 & 0 & 3 \end{pmatrix} \\
\xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} & \begin{pmatrix} 1 & -2 & \frac{1}{3} & 1 & \frac{4}{3} \\ 0 & 0 & -\frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & \frac{5}{3} & -1 & \frac{5}{3} \end{pmatrix} \\
\xrightarrow{R_2 \to -3R_2} & \begin{pmatrix} 1 & -2 & \frac{1}{3} & 1 & \frac{4}{3} \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & \frac{5}{3} & -1 & \frac{5}{3} \end{pmatrix} \\
\xrightarrow[R_3 \to R_3 - \frac{5}{3}R_2]{R_1 \to R_1 - \frac{1}{3}R_2} & \begin{pmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & -1 & 5 \end{pmatrix} \\
\xrightarrow{R_3 \to \frac{5}{3}R_3} & \begin{pmatrix} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix} \\
\xrightarrow{R_1 \to R_1 - R_3} & \begin{pmatrix} 1 & -2 & 0 & 0 & 7 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -5 \end{pmatrix}
\end{aligned}$$

We have arrived at the reduced row echelon form of A.

- The set
$$\{(2,1,0,0,0)^T, (-7,0,2,5,1)^T\}$$
constitutes a basis for $\text{Null}(\mathbf{A})$.
- The set
$$\{(1,-2,0,0,7), (0,0,1,0,-2), (0,0,0,1,-5)\}$$
constitutes a basis for $\text{Row}(\mathbf{A})$.
- The first, third and fourth columns contain leading entries. Therefore the set
$$\{(3,1,1)^T, (1,0,2)^T, (3,1,0)^T\}$$
constitutes a basis for $\text{Col}(\mathbf{A})$.

□

Exercise 3.5 Question 1(d)

Find a basis for the null space, a basis for the row space and a basis for the column space for the given matrices.

$$\begin{pmatrix} 1 & 1 & -1 & 7 \\ 1 & 4 & 5 & 16 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{pmatrix}$$

Solution: By Gaussian elimination, we have

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 & -1 & 7 \\ 1 & 4 & 5 & 16 \\ 1 & 3 & 3 & 13 \\ 2 & 5 & 4 & 23 \end{pmatrix} \\
&\xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - 2R_1}} \begin{pmatrix} 1 & 1 & -1 & 7 \\ 0 & 3 & 6 & 9 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{pmatrix} \\
&\xrightarrow{R_2 \to \frac{1}{3}R_2} \begin{pmatrix} 1 & 1 & -1 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 4 & 6 \\ 0 & 3 & 6 & 9 \end{pmatrix} \\
&\xrightarrow{\substack{R_1 \to R_1 - R_2 \\ R_3 \to R_3 - 2R_2 \\ R_4 \to R_4 - 3R_2}} \begin{pmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

We have arrived at the reduced row echelon form of A.

- The set
$$\{(3, -2, 1, 0)^T, (-4, -3, 0, 1)^T\}$$
constitutes a basis for Null($\mathbf{A}$).
- The set
$$\{(1, 0, -3, 4), (0, 1, 2, 3)\}$$
constitutes a basis for Row($\mathbf{A}$).
- The first and second columns contain leading entries. Therefore the set
$$\{(1, 1, 1, 2)^T, (1, 4, 3, 5)^T\}$$
constitutes a basis for Col($\mathbf{A}$).

□