Definition

Let $f(x)$ be a continuous function. A **primitive function**, or an **anti-derivative**, of $f(x)$ is a function $F(x)$ such that

$$F'(x) = f(x).$$

The collection of all anti-derivatives of $f(x)$ is called the **indefinite integral** of $f(x)$ and is denoted by

$$\int f(x)dx.$$

The function $f(x)$ is called the **integrand** of the integral.

Note: Anti-derivative of a function is not unique. If $F(x)$ is an anti-derivative of f, then $F(x) + C$ is an anti-derivative of $f(x)$ for any constant C. Moreover, any anti-derivative of $f(x)$ is of the form $F(x) + C$ and we write

$$\int f(x)dx = F(x) + C$$

where C is arbitrary constant called the **integration constant**. Note that $\int f(x)dx$ is not a single function but a collection of functions.

Theorem

Let $f(x)$ and $g(x)$ be continuous functions and k be a constant.

1. $\int (f(x) + g(x))dx = \int f(x)dx + \int g(x)dx$
2. $\int kf(x)dx = k \int f(x)dx$

Theorem (formulas for indefinite integrals)

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq 1$$

$$\int e^x dx = e^x + C; \qquad \int \frac{1}{x} dx = \ln |x| + C$$

$$\int \cos x dx = \sin x + C; \qquad \int \sin x dx = -\cos x + C$$

$$\int \sec^2 x dx = \tan x + C; \qquad \int \csc^2 x dx = -\cot x + C$$

$$\int \sec x \tan x dx = \sec x + C; \qquad \int \csc x \cot x dx = -\csc x + C$$

Example

1. $$\int (x^3 - x + 5)dx = \frac{x^4}{4} - \frac{x^2}{2} + 5x + C$$

2. $$\begin{aligned}\int \frac{(x+1)^2}{x}dx &= \int \frac{x^2+2x+1}{x}dx \\ &= \int \left(x + 2 + \frac{1}{x}\right) dx \\ &= \frac{x^2}{2} + 2x + \ln|x| + C\end{aligned}$$

3. $$\begin{aligned}\int \frac{3x^2+\sqrt{x}-1}{\sqrt{x}}dx &= \int \left(3x^{3/2} + 1 - x^{-1/2}\right) dx \\ &= \frac{6}{5}x^{\frac{5}{2}} + x - 2x^{\frac{1}{2}} + C\end{aligned}$$

4. $$\begin{aligned}\int \left(\frac{3\sin x}{\cos^2 x} - 2e^x\right) dx &= \int (3\sec x \tan x - 2e^x)\, dx \\ &= 3\sec x - 2e^x + C\end{aligned}$$

Example

Suppose we want to compute

$$\int x\sqrt{x^2+4}\,dx$$

First we let

$$u = x^2 + 4.$$

Subsequently we may formally write

$$du = \frac{du}{dx}\,dx = \left[\frac{d}{dx}(x^2+4)\right]dx = 2xdx$$

Here du is called the differential of u defined as $\dfrac{du}{dx}\,dx$. Thus the integral is

$$\begin{aligned}\int x\sqrt{x^2+4}\,dx &= \frac{1}{2}\int \sqrt{x^2+4}(2xdx) = \frac{1}{2}\int \sqrt{u}\,du \\ &= \frac{u^{\frac{3}{2}}}{3} + C = \frac{(x^2+4)^{\frac{3}{2}}}{3} + C\end{aligned}$$

Example

$$\begin{aligned}
\int x\sqrt{x^2+4}\,dx &= \int \sqrt{x^2+4}\,d\left(\frac{x^2}{2}\right) \\
&= \frac{1}{2}\int \sqrt{x^2+4}\,dx^2 \\
&= \frac{1}{2}\int \sqrt{x^2+4}\,d(x^2+4) \\
&= \frac{(x^2+4)^{\frac{3}{2}}}{3} + C
\end{aligned}$$

Theorem

Let $f(x)$ be a continuous function defined on $[a, b]$. Suppose there exists a differentiable function $u = \varphi(x)$ and continuous function $g(u)$ such that $f(x) = g(\varphi(x))\varphi'(x)$ for any $x \in (a, b)$. Then

$$\begin{aligned}\int f(x)dx &= \int g(\varphi(x))\varphi'(x)dx \\ &= \int g(u)du\end{aligned}$$

Example

$$\int x^2 e^{x^3+1} dx$$

Let $u = x^3 + 1$,

then $du = 3x^2 dx$

$$= \frac{1}{3}\int e^u du$$

$$= \frac{e^u}{3} + C$$

$$= \frac{e^{x^3+1}}{3} + C$$

$$\int x^2 e^{x^3+1} dx$$

$$= \int e^{x^3+1} d\left(\frac{x^3}{3}\right)$$

$$= \frac{1}{3}\int e^{x^3+1} dx^3$$

$$= \frac{1}{3}\int e^{x^3+1} d(x^3+1)$$

$$= \frac{e^{x^3+1}}{3} + C$$

Example

$$\int \cos^4 x \sin x dx$$

Let $u = \cos x$,

then $du = -\sin x dx$

$$= -\int u^4 du$$

$$= -\frac{u^5}{5} + C$$

$$= -\frac{\cos^5 x}{5} + C$$

$$\int \cos^4 x \sin x dx$$

$$= \int \cos^4 x d(-\cos x)$$

$$= -\int \cos^4 x d \cos x$$

$$= -\frac{\cos^5 x}{5} + C$$

Example

$$\int \frac{dx}{x \ln x}$$

Let $u = \ln x$,

then $du = \dfrac{dx}{x}$

$$= \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\ln x| + C$$

$$\int \frac{dx}{x \ln x}$$

$$= \int \frac{d \ln x}{\ln x}$$

$$= \ln|\ln x| + C$$

Example

$$\int \frac{dx}{e^x+1}$$

Let $u = 1 + e^{-x}$,

then $du = -e^{-x}dx$

$$\begin{aligned}
&= \int \frac{e^{-x}dx}{1+e^{-x}} \\
&= -\int \frac{du}{u} \\
&= -\ln u + C \\
&= -\ln(1+e^{-x}) + C \\
&= x - \ln(1+e^x) + C
\end{aligned}$$

$$\begin{aligned}
&\int \frac{dx}{e^x+1} \\
&= \int \left(1 - \frac{e^x}{1+e^x}\right) dx \\
&= x - \int \frac{de^x}{1+e^x} \\
&= x - \ln(1+e^x) + C
\end{aligned}$$

Example

$$\int \frac{dx}{1+\sqrt{x}}$$

Let $u = 1+\sqrt{x}$,

then $du = \dfrac{dx}{2\sqrt{x}}$

$$= 2\int \frac{(u-1)du}{u}$$

$$= 2\int \left(1-\frac{1}{u}\right) du$$

$$= 2u - 2\ln u + C'$$

$$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$$

$$\int \frac{dx}{1+\sqrt{x}}$$

$$= \int \frac{\sqrt{x}\,dx}{\sqrt{x}(1+\sqrt{x})}$$

$$= 2\int \frac{\sqrt{x}\,d\sqrt{x}}{1+\sqrt{x}}$$

$$= 2\int \left(1-\frac{1}{1+\sqrt{x}}\right) d\sqrt{x}$$

$$= 2\sqrt{x} - 2\ln(1+\sqrt{x}) + C$$

Definition

Let $f(x)$ be a function on $[a, b]$.

1. A **Partition** of $[a, b]$ is a set of finite points

$$P = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$$

and we define

$$\begin{aligned} \Delta x_k &= x_k - x_{k-1}, \text{ for } k = 1, 2, \ldots, n \\ \|P\| &= \max_{1 \le k \le n} \{\Delta x_k\} \end{aligned}$$

2. The **lower** and **upper Riemann sums** with respect to partition P are

$$\mathcal{L}(f, P) = \sum_{k=1}^{n} m_k \Delta x_k, \text{ and } \mathcal{U}(f, P) = \sum_{k=1}^{n} M_k \Delta x_k$$

where

$$m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}, \text{ and } M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$$

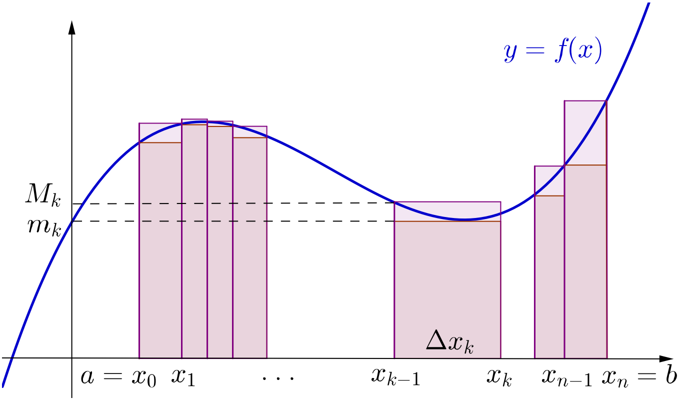


Figure: Upper and lower Riemann sum

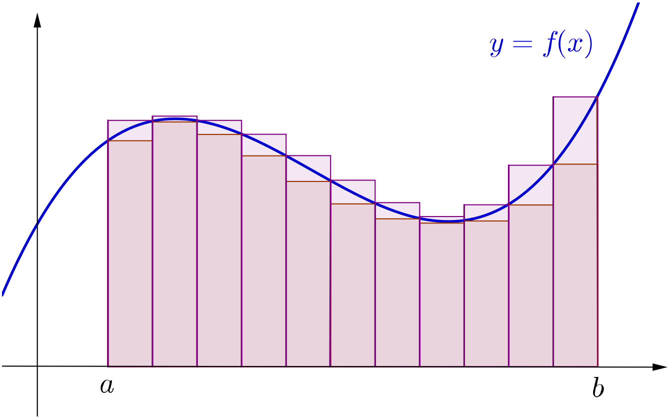


Figure: Upper and lower Riemann sum

Definition (Riemann integral)

Let $[a, b]$ be a closed and bounded interval and $f : [a, b] \to \mathbb{R}$ be a real valued function defined on $[a, b]$. We say that $f(x)$ is **Riemann integrable** on $[a, b]$ if the limits of $\mathcal{L}(f, P)$ and $\mathcal{U}(f, P)$ *exist* as $\|P\|$ tends to 0 and are *equal*. In this case, we define the **Riemann integral** of $f(x)$ over $[a, b]$ by

$$\int_a^b f(x)dx = \lim_{\|P\|\to 0} \mathcal{L}(f, P) = \lim_{\|P\|\to 0} \mathcal{U}(f, P).$$

Note: We say that $\lim_{\|P\|\to 0} \mathcal{L}(f, P) = L$ if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|P\| < \delta$, then $|\mathcal{L}(f, P) - L| < \varepsilon$.

Theorem

Let $f(x)$ and $g(x)$ be integrable functions on $[a, b]$, $a < c < b$ and k be constants.

1. $\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
2. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
4. $\int_b^a f(x)dx = -\int_a^b f(x)dx$

Example

Let $f(x)$ be the Dirichlet's function defined by

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Then $f(x)$ is not Riemann integrable on $[0,1]$. In fact, for any partition $P = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$, we have

$$\inf\{f(x) : x_{k-1} \le x \le x_k\} = 0, \text{ and } \sup\{f(x) : x_{k-1} \le x \le x_k\} = 1.$$

Thus we have

$$\mathcal{L}(f,P) = \sum_{k=1}^{n} 0 \cdot \Delta x_k = 0, \text{ and } \mathcal{U}(f,P) = \sum_{k=1}^{n} 1 \cdot \Delta x_k = 1.$$

Therefore the limits $\lim\limits_{\|P\|\to 0} \mathcal{L}(f,P)$ and $\lim\limits_{\|P\|\to 0} \mathcal{U}(f,P)$ exist but are not equal. Therefore $f(x)$ is not Riemann integrable on $[0,1]$.

Theorem

Suppose $f(x)$ is a continuous function on $[a, b]$. Then $f(x)$ is Riemann integrable on $[a, b]$ and we have

$$\begin{aligned}\int_a^b f(x)dx &= \lim_{n\to\infty}\sum_{k=1}^{n} f(x_k)\Delta x_k \\ &= \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a+\frac{k}{n}(b-a)\right)\left(\frac{b-a}{n}\right).\end{aligned}$$

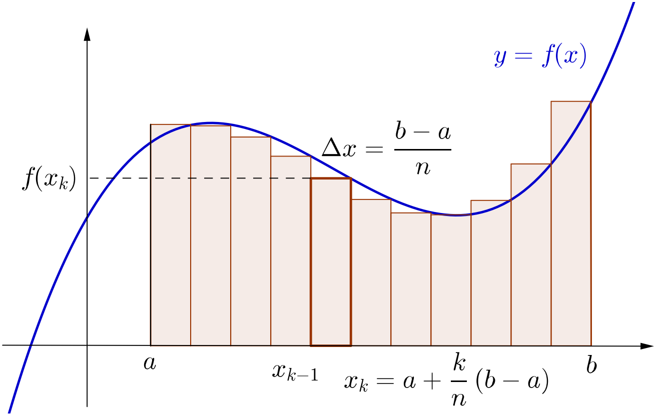


Figure: Formula for Riemann integral

Example

Use the formula for definite integral of continuous function to evaluate

$$\int_0^1 x^2 dx$$

Solution

$$\begin{aligned}
\int_0^1 x^2 dx &= \lim_{n\to\infty} \sum_{k=1}^{n} \left(0 + \frac{k}{n}(1-0)\right)^2 \left(\frac{1-0}{n}\right) \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^2}{n^3} \\
&= \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} \\
&= \frac{1}{3}
\end{aligned}$$

Example

Use the formula for definite integral of continuous function to evaluate

$$\int_0^1 e^x dx$$

Solution

$$\begin{aligned}
\int_0^1 e^x dx &= \lim_{n\to\infty}\left(f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+f\left(\frac{3}{n}\right)+\cdots+f\left(\frac{n}{n}\right)\right)\left(\frac{1-0}{n}\right) \\
&= \lim_{n\to\infty}\left(e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n}{n}}\right)\left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty}\frac{e^{\frac{1}{n}}((e^{\frac{1}{n}})^n-1)}{(e^{\frac{1}{n}}-1)n} \\
&= \lim_{n\to\infty} e^{\frac{1}{n}}(e-1)\lim_{x\to 0}\frac{x}{e^x-1} \\
&= e-1
\end{aligned}$$

Theorem (Fundamental theorem of calculus)

Let $f(x)$ be a function which is continuous on $[a, b]$.
First part: *Let $F : [a, b] \to \mathbb{R}$ be the function defined by*

$$F(x) = \int_a^x f(t)dt$$

Then $F(x)$ is continuous on $[a, b]$, differentiable on (a, b) and

$$F'(x) = f(x)$$

for any $x \in (a, b)$. Put in another way, we have

$$\frac{d}{dx} \int_a^x f(t)dt = f(x) \quad \text{for } x \in (a, b)$$

Second part: *Let $F(x)$ be a primitive function of $f(x)$, in other words, $F(x)$ is a continuous function on $[a, b]$ and $F'(x) = f(x)$ for any $x \in (a, b)$. Then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

Example

Let $f(x) = \sqrt{1-x^2}$. The graph of $y = f(x)$ is a unit semicircle centered at the origin. Using the formula for area of circular sectors, we calculate

$$F(x) = \int_0^x f(t)dt = \int_0^x \sqrt{1-t^2}dt = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2}.$$

By fundamental theorem of calculus, we know that $F(x)$ is an anti-derivative of $f(x)$. One may check this from direct calculation

$$\begin{aligned} F'(x) &= \frac{1}{2}\left(\sqrt{1-x^2} - \frac{x^2}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\right) \\ &= \frac{1}{2}\left(\frac{1-x^2-x^2+1}{\sqrt{1-x^2}}\right) \\ &= \sqrt{1-x^2} \\ &= f(x) \end{aligned}$$

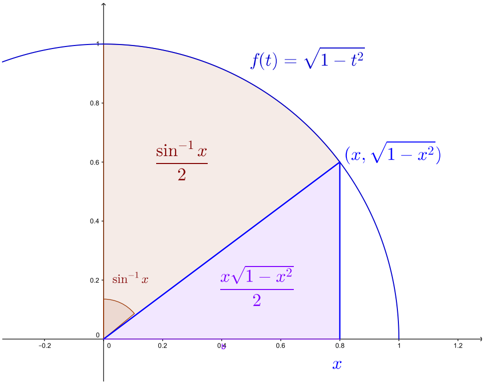


Figure: $\displaystyle\int_0^x \sqrt{1-t^2}\,dt = \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1} x}{2}$

Example

1. $$\int_1^3 (x^3 - 4x + 5)dx = \left[\frac{x^4}{4} - 2x^2 + 5x\right]_1^3 = \left[\left(\frac{3^4}{4} - 2(3^2) + 5(3)\right) - \left(\frac{1^4}{4} - 2(1^2) + 5(1)\right)\right] = 14$$

2. $$\int_0^{\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}dx = 2\int_0^{\pi^2} \sin\sqrt{x}\, d\sqrt{x} = 2\left[-\cos\sqrt{x}\right]_0^{\pi^2} = 2\left[-\cos\sqrt{\pi^2} - (-\cos 0)\right] = 4$$

3. $$\int_3^5 x\sqrt{x^2-9}\,dx = \frac{1}{2}\int_3^5 \sqrt{x^2-9}\,d(x^2-9) = \frac{1}{3}\left[(x^2-9)^{\frac{3}{2}}\right]_3^5 = \frac{64}{3}$$

Example

We have the following formulas for derivatives of functions defined by integrals.

1. $$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$
2. $$\frac{d}{dx}\int_x^b f(t)dt = -f(x)$$
3. $$\frac{d}{dx}\int_a^{v(x)} f(t)dt = f(v)\frac{dv}{dx}$$
4. $$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$$

Proof.

1. This is the first part of fundamental theorem of calculus.
2. $$\begin{aligned}\frac{d}{dx}\int_x^b f(t)dt &= \frac{d}{dx}\left(-\int_b^x f(t)dt\right)\\ &= -f(x)\end{aligned}$$
3. $$\begin{aligned}\frac{d}{dx}\int_a^{v(x)} f(t)dt &= \left(\frac{d}{dv}\int_a^{v(x)} f(t)dt\right)\frac{dv}{dx}\\ &= f(v)\frac{dv}{dx}\end{aligned}$$
4. $$\begin{aligned}\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt &= \frac{d}{dx}\left(\int_c^{v(x)} f(t)dt + \int_{u(x)}^{c} f(t)dt\right)\\ &= \frac{d}{dx}\left(\int_c^{v(x)} f(t)dt - \int_c^{u(x)} f(t)dt\right)\\ &= f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}\end{aligned}$$

□

Example

Find $F'(x)$ for the the functions.

1. $F(x) = \int_1^x \sqrt{t}e^t dt$
2. $F(x) = \int_x^\pi \frac{\sin t}{t} dt$
3. $F(x) = \int_0^{\sin x} \sqrt{1 + t^4} dt$
4. $F(x) = \int_{-x}^{x^2} e^{t^2} dt$

Solution

1. $$\frac{d}{dx}\int_1^x \sqrt{t}e^t dt = \sqrt{x}e^x$$

2. $$\frac{d}{dx}\int_x^\pi \frac{\sin t}{t} dt = -\frac{\sin x}{x}$$

3. $$\begin{aligned}\frac{d}{dx}\int_0^{\sin x} \sqrt{1+t^4}dt &= \sqrt{1+\sin^4 x}\frac{d}{dx}\sin x \\ &= \cos x\sqrt{1+\sin^4 x}\end{aligned}$$

4. $$\begin{aligned}\frac{d}{dx}\int_{-x}^{x^2} e^{t^2} dt &= e^{(x^2)^2}\frac{d}{dx}x^2 - e^{(-x)^2}\frac{d}{dx}(-x) \\ &= 2xe^{x^4} + e^{x^2}\end{aligned}$$

Techniques

When we evaluate integrals which involve trigonometric functions, the following trigonometric identities are very useful.

1.
   - $\cos^2 x + \sin^2 x = 1$
   - $\sec^2 x = 1 + \tan^2 x$
   - $\csc^2 x = 1 + \cot^2 x$
2.
   - $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
   - $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
   - $\cos x \sin x = \dfrac{\sin 2x}{2}$
3.
   - $\cos x \cos y = \frac{1}{2}(\cos(x + y) + \cos(x - y))$
   - $\cos x \sin y = \frac{1}{2}(\sin(x + y) - \sin(x - y))$
   - $\sin x \sin y = \frac{1}{2}(\cos(x - y) - \cos(x + y))$

Techniques

To evaluate

$$\int \cos^m x \sin^n x dx$$

where m, n are non-negative integers,

- *Case 1. If m is odd, use $\cos x dx = d \sin x$. (Substitute $u = \sin x$.)*
- *Case 2. If n is odd, use $\sin x dx = -d \cos x$. (Substitute $u = \cos x$.)*
- *Case 3. If both m, n are even, then use double angle formulas to reduce the power.*

$$\begin{aligned} \cos^2 x &= \frac{1 + \cos 2x}{2} \\ \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \cos x \sin x &= \frac{\sin 2x}{2} \end{aligned}$$

Techniques

1. $\displaystyle\int \tan x dx = \ln|\sec x| + C$
2. $\displaystyle\int \cot x dx = \ln|\sin x| + C$
3. $\displaystyle\int \sec x dx = \ln|\sec x + \tan x| + C$
4. $\displaystyle\int \csc x dx = \ln|\csc x - \cot x| + C$

Proof

We prove (1), (3) and the rest are left as exercise.

1. $$\begin{aligned}\int \tan x dx &= \int \frac{\sin x dx}{\cos x} \\ &= -\int \frac{d\cos x}{\cos x} \\ &= -\ln|\cos x| + C \\ &= \ln|\sec x| + C\end{aligned}$$

3. $$\begin{aligned}\int \sec x dx &= \int \frac{\sec x(\sec x + \tan x)dx}{(\sec x + \tan x)} \\ &= \int \frac{(\sec^2 x + \sec x \tan x)dx}{(\sec x + \tan x)} \\ &= \int \frac{d(\tan x + \sec x)}{(\sec x + \tan x)} \\ &= \ln|\sec x + \tan x| + C\end{aligned}$$

Techniques

To evaluate

$$\int \sec^m x \tan^n x dx$$

where m, n *are non-negative integers,*

- *Case 1. If* m *is even, use* $\sec^2 x dx = d \tan x$. *(Substitute* $u = \tan x$.*)*
- *Case 2. If* n *is odd, use* $\sec x \tan x dx = d \sec x$. *(Substitute* $u = \sec x$.*)*
- *Case 3. If both* m *is odd and* n *is even, use* $\tan^2 x = \sec^2 x - 1$ *to write everything in terms of* $\sec x$.

Example

Evaluate the following integrals.

1. $\displaystyle\int \sin^2 x dx$
2. $\displaystyle\int \cos^4 3x dx$
3. $\displaystyle\int \cos 2x \cos x dx$
4. $\displaystyle\int \cos 3x \sin 5x dx$

Solution

1. $$\int \sin^2 x dx = \int \left(\frac{1-\cos 2x}{2}\right) dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

2. $$\begin{aligned} \int \cos^4 x dx &= \int \left(\frac{1+\cos 2x}{2}\right)^2 dx \\ &= \int \left(\frac{1+2\cos 2x+\cos^2 2x}{4}\right) dx \\ &= \frac{x}{4} + \frac{\sin 2x}{4} + \int \left(\frac{1+\cos 4x}{8}\right) dx \\ &= \frac{3x}{8} + \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C \end{aligned}$$

3. $$\int \cos 2x \cos x dx = \frac{1}{2}\int (\cos 3x + \cos x)\, dx = \frac{\sin 3x}{6} + \frac{\sin x}{2} + C$$

4. $$\int \cos 3x \sin 5x dx = \frac{1}{2}\int (\sin 8x + \sin 2x)\, dx = -\frac{\cos 8x}{16} - \frac{\cos 2x}{4} + C$$

Example

Evaluate the following integrals.

1. $\displaystyle\int \cos x \sin^4 x dx$
2. $\displaystyle\int \cos^2 x \sin^3 x dx$
3. $\displaystyle\int \cos^4 x \sin^2 x dx$

Solution

$$
\begin{aligned}
1.\ \int \cos x \sin^4 x dx &= \int \sin^4 x d\sin x = \frac{\sin^5 x}{5} + C \\
2.\ \int \cos^2 x \sin^3 x dx &= -\int \cos^2 x(1 - cos^2 x) d\cos x \\
&= -\int (\cos^2 x - cos^4 x) d\cos x \\
&= -\frac{cos^3 x}{3} + \frac{cos^5 x}{5} C \\
3.\ \int \cos^4 x \sin^2 x dx &= \int \left(\frac{1+\cos 2x}{2}\right)\left(\frac{\sin 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int \left(\sin^2 2x + \cos 2x \sin^2 2x\right) dx \\
&= \frac{1}{8}\int \left(\frac{1-\cos 4x}{2}\right) dx + \frac{1}{16}\int \sin^2 2x d\sin 2x \\
&= \frac{x}{16} - \frac{\sin 4x}{64} + \frac{\sin^3 2x}{48} + C
\end{aligned}
$$

Example

Evaluate the following integrals.

1. $\int \sec^2 x \tan^2 x dx$
2. $\int \sec x \tan^3 x dx$
3. $\int \tan^3 x dx$

Solution

1. $$\int \sec^2 x \tan^2 x dx = \int \tan^2 x d\tan x = \frac{\tan^3 x}{3} + C$$

2. $$\begin{aligned}\int \sec x \tan^3 x dx &= \int \tan^2 x d\sec x = \int (\sec^2 x - 1) d\sec x \\ &= \frac{\sec^3 x}{3} - \sec x + C\end{aligned}$$

3. $$\begin{aligned}\int \tan^3 x dx &= \int \tan x(\sec^2 x - 1)dx \\ &= \int \tan x \sec^2 x dx - \int \tan x dx \\ &= \int \tan x d\tan x - \ln|\sec x| \\ &= \frac{\tan^2 x}{2} - \ln|\sec x| + C\end{aligned}$$

Techniques

Suppose the integrand is of the form $u(x)v'(x)$. Then we may evaluate the integration using the formula

$$\int uv'dx = uv - \int u'vdx.$$

The above formula is called integration by parts. It is usually written in the form

$$\int udv = uv - \int vdu.$$

Example

Evaluate the following integrals.

1. $\int xe^{3x}\,dx$
2. $\int x^2 \cos x dx$
3. $\int x^3 \ln x dx$
4. $\int \ln x dx$

Solution

1. $$\int xe^{3x}dx = \frac{1}{3}\int xde^{3x} = \frac{xe^{3x}}{3} - \frac{1}{3}\int e^{3x}dx = \frac{xe^{3x}}{3} - \frac{e^{3x}}{9} + C$$

2. $$\begin{aligned}\int x^2 \cos x dx &= \int x^2 d \sin x \\ &= x^2 \sin x - \int \sin x dx^2 \\ &= x^2 \sin x - 2\int x \sin x dx \\ &= x^2 \sin x + 2\int x d \cos x \\ &= x^2 \sin x + 2x \cos x - 2\int \cos x dx \\ &= x^2 \sin x + 2x \cos x - 2 \sin x + C\end{aligned}$$

Solution

$$\begin{aligned}
3.\ \int x^3 \ln x dx &= \frac{1}{4}\int \ln x dx^4 \\
&= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^4 d\ln x \\
&= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^4 \left(\frac{1}{x}\right) dx \\
&= \frac{x^4 \ln x}{4} - \frac{1}{4}\int x^3 dx \\
&= \frac{x^4 \ln x}{4} - \frac{x^4}{16} + C
\end{aligned}$$

$$\begin{aligned}
4.\ \int \ln x dx &= x\ln x - \int x d\ln x \\
&= x\ln x - \int dx \\
&= x\ln x - x + C
\end{aligned}$$

Example

Evaluate the following integrals.

1. $\int \sin^{-1} x dx$
2. $\int \ln(1+x^2) dx$
3. $\int \sec^3 x dx$
4. $\int e^x \sin x dx$

Solution

1.
$$\begin{aligned}\int \sin^{-1} x dx &= x \sin^{-1} x - \int x d \sin^{-1} x \\ &= x \sin^{-1} x - \int \frac{x dx}{\sqrt{1-x^2}} \\ &= x \sin^{-1} x + \frac{1}{2} \int \frac{d(1-x^2)}{\sqrt{1-x^2}} \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C\end{aligned}$$

2.
$$\begin{aligned}\int \ln(1+x^2) dx &= x \ln(1+x^2) - \int x d \ln(1+x^2) \\ &= x \ln(1+x^2) - 2 \int \frac{x^2 dx}{1+x^2} \\ &= x \ln(1+x^2) - 2 \int \left(1 - \frac{1}{1+x^2}\right) dx \\ &= x \ln(1+x^2) - 2x + 2 \tan^{-1} x + C\end{aligned}$$

Solution

3.
$$\begin{aligned}
\int \sec^3 x dx &= \int \sec x d \tan x \\
&= \sec x \tan x - \int \tan x d \sec x \\
&= \sec x \tan x - \int \sec x \tan^2 x dx \\
&= \sec x \tan x - \int \sec x (\sec^2 x - 1) dx \\
&= \sec x \tan x - \int \sec^3 x dx + \int \sec x dx \\
2 \int \sec^3 x dx &= \sec x \tan x + \int \sec x dx \\
\int \sec^3 x dx &= \frac{\sec x \tan x + \ln |\sec x + \tan x|}{2} + C
\end{aligned}$$

Solution

4.
$$\begin{aligned}
\int e^x \sin x dx &= \int \sin x de^x \\
&= e^x \sin x - \int e^x d \sin x \\
&= e^x \sin x - \int e^x \cos x dx \\
&= e^x \sin x - \int \cos x de^x \\
&= e^x \sin x - e^x \cos x + \int e^x d \cos x \\
&= e^x \sin x - e^x \cos x - \int e^x \sin x dx \\
2\int e^x \sin x dx &= e^x \sin x - e^x \cos x + C' \\
\int e^x \sin x dx &= \frac{1}{2}(e^x \sin x - e^x \cos x) + C
\end{aligned}$$

Techniques

For integral of the forms

$$\begin{aligned} I_n = & \int \cos^n x dx, \int \sin^n x dx, \int x^n \cos x dx, \int x^n \sin x dx, \\ & \int \sec^n x dx, \int \csc^n x dx, \int x^n e^x dx, \int (\ln x)^n dx, \\ & \int e^x \cos^n x dx, \int e^x \sin^n x dx, \int \frac{dx}{(x^2+a^2)^n}, \int \frac{dx}{(a^2-x^2)^n}, \end{aligned}$$

we may use integration by parts to find a formula to express I_n in terms of I_k with $k < n$. Such a formula is called reduction formula.

Example

Let

$$I_n = \int x^n \cos x dx$$

for positive integer n. Prove that

$$I_n = x^n \sin x + nx^{n-1} \cos x - n(n-1)I_{n-2}, \text{ for } n \geq 2$$

Proof.

$$\begin{aligned} I_n = \int x^n \cos x dx &= \int x^n d \sin x \\ &= x^n \sin x - \int \sin x dx^n \\ &= x^n \sin x - n \int x^{n-1} \sin x dx \\ &= x^n \sin x + n \int x^{n-1} d \cos x \\ &= x^n \sin x + n x^{n-1} \cos x - n \int \cos x dx^{n-1} \\ &= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x dx \\ &= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2} \end{aligned}$$

□

Example

Let

$$I_n = \int \frac{dx}{x^2 + a^2}$$

where $a > 0$ is a positive real number for positive integer n. Prove that

$$I_n = \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)} I_{n-1}, \quad \text{for } n \geq 2$$

Proof

$$\begin{aligned}
I_n = \int \frac{dx}{(x^2+a^2)^n} &= \frac{x}{(x^2+a^2)^n} - \int x d\left(\frac{1}{(x^2+a^2)^n}\right) \\
&= \frac{x}{(x^2+a^2)^n} + \int \frac{2nx^2 dx}{(x^2+a^2)^{n+1}} \\
&= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{(x^2+a^2-a^2)dx}{(x^2+a^2)^{n+1}} \\
&= \frac{x}{(x^2+a^2)^n} + 2n \int \frac{dx}{(x^2+a^2)^n} - 2na^2 \int \frac{dx}{(x^2+a^2)^{n+1}} \\
&= \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1} \\
I_{n+1} &= \frac{x}{2na^2(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n
\end{aligned}$$

Replacing n by $n-1$, we have

$$I_n = \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2} I_{n-1}.$$

Alternative proof.

$$
\begin{aligned}
I_n &= \frac{1}{a^2}\int \frac{x^2+a^2-x^2}{(x^2+a^2)^n}dx \\
&= \frac{1}{a^2}\int \left(\frac{1}{(x^2+a^2)^{n-1}} - \frac{x^2}{(x^2+a^2)^n}\right) dx \\
&= \frac{1}{a^2}I_{n-1} - \frac{1}{2a^2}\int \frac{x}{(x^2+a^2)^n}d(x^2+a^2) \\
&= \frac{1}{a^2}I_{n-1} + \frac{1}{2(n-1)a^2}\int xd\left(\frac{1}{(x^2+a^2)^{n-1}}\right) \\
&= \frac{1}{a^2}I_{n-1} + \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} - \frac{1}{2(n-1)a^2}\int \frac{dx}{(x^2+a^2)^{n-1}} \\
&= \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \left(\frac{1}{a^2} - \frac{1}{2(n-1)a^2}\right) I_{n-1} \\
&= \frac{x}{2(n-1)a^2(x^2+a^2)^{n-1}} + \frac{2n-3}{2(n-1)a^2}I_{n-1}
\end{aligned}
$$

□

Example

Prove the following reduction formula

$$\int \sin^n x dx = -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx$$

for $n \geq 2$. Hence show that

$$\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \dfrac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3} & \text{when } n \text{ is odd} \\ \dfrac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\cdot\dfrac{\pi}{2} & \text{when } n \text{ is even} \end{cases}$$

Proof

$$\begin{aligned}
\int \sin^n x dx &= -\int \sin^{n-1} x d\cos x \\
&= -\cos x \sin^{n-1} x + \int \cos x d\sin^{n-1} x \\
&= -\cos x \sin^{n-1} x + (n-1)\int \cos^2 x \sin^{n-2} x dx \\
&= -\cos x \sin^{n-1} x + (n-1)\int (1-\sin^2 x)\sin^{n-2} x dx \\
n\int \sin^n x dx &= -\cos x \sin^{n-1} x + (n-1)\int \sin^{n-2} x dx \\
\int \sin^n x dx &= -\frac{1}{n}\cos x \sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x dx
\end{aligned}$$

Proof

Hence when n is odd

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^n x dx &= -\left[\frac{1}{n}\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\ &= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\frac{\pi}{2}} \sin^{n-4} x dx \\ &\vdots \\ &= \frac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3}\int_0^{\frac{\pi}{2}} \sin x dx \\ &= \frac{(n-1)\cdot(n-3)\cdots 6\cdot 4\cdot 2}{n\cdot(n-2)\cdots 7\cdot 5\cdot 3}\end{aligned}$$

Proof.

when n is even

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \sin^n x dx &= -\left[\frac{1}{n}\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\
&= \frac{n-1}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x dx \\
&= \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\int_0^{\frac{\pi}{2}} \sin^{n-4} x dx \\
&\vdots \\
&= \frac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\int_0^{\frac{\pi}{2}} dx \\
&= \frac{(n-1)\cdot(n-3)\cdots 7\cdot 5\cdot 3}{n\cdot(n-2)\cdots 6\cdot 4\cdot 2}\cdot\frac{\pi}{2}
\end{aligned}$$

□

Example

$$I_n = \int x^n e^x dx; \qquad I_n = x^n e^x - nI_{n-1}, \;\; n \geq 1$$

$$I_n = \int (\ln x)^n dx; \qquad I_n = x(\ln x)^n - nI_{n-1}, \;\; n \geq 1$$

$$I_n = \int x^n \sin x dx; \qquad I_n = -x^n \cos x + nx^{n-1} \sin x - n(n-1)I_{n-2}, \;\; n \geq 2$$

$$I_n = \int \cos^n x dx; \qquad I_n = \frac{\cos^{n-1} x \sin x}{n} + (n-1)I_{n-2}, \;\; n \geq 2$$

$$I_n = \int \sec^n x dx; \qquad I_n = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} I_{n-2}, \;\; n \geq 2$$

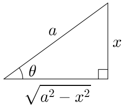
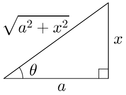
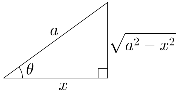
$$I_n = \int e^x \cos^n x dx; \qquad I_n = \frac{e^x \cos^{n-1} x(\cos x + n \sin x)}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2}, \;\; n \geq 2$$

$$I_n = \int e^x \sin^n x dx; \qquad I_n = \frac{e^x \sin^{n-1} x(\sin x - n \cos x)}{n^2+1} + \frac{n(n-1)}{n^2+1} I_{n-2}, \;\; n \geq 2$$

$$I_n = \int x^n \sqrt{x+a} dx; \qquad I_n = \frac{2x^n (x+a)^{\frac{3}{2}}}{2n+3} - \frac{2na}{2n+3} I_{n-1}, \;\; n \geq 1$$

$$I_n = \int \frac{x^n}{\sqrt{x+a}} dx; \qquad I_n = \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2na}{2n+1} I_{n-1}, \;\; n \geq 1$$

Techniques (Trigonometric substitution)

Expression	Substitution	dx	Trigonometric ratios	
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$dx = a\cos\theta d\theta$		$\cos\theta = \dfrac{\sqrt{a^2-x^2}}{a}$ $\sin\theta = \dfrac{x}{a}$ $\tan\theta = \dfrac{a}{\sqrt{a^2-x^2}}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$dx = a\sec^2\theta d\theta$		$\cos\theta = \dfrac{a}{\sqrt{a^2+x^2}}$ $\sin\theta = \dfrac{x}{\sqrt{a^2+x^2}}$ $\tan\theta = \dfrac{x}{a}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$dx = a\sec\theta\tan\theta d\theta$		$\cos\theta = \dfrac{x}{a}$ $\sin\theta = \dfrac{\sqrt{x^2-a^2}}{a}$ $\tan\theta = \dfrac{\sqrt{a^2-x^2}}{x}$

Theorem

1. $$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$
2. $$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$
3. $$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \cos^{-1}\frac{a}{x} + C$$

Proof

1. *Let* $x = a\sin\theta$. *Then*

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2\sin^2\theta} = a\cos\theta \\ dx &= a\cos\theta d\theta\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{1}{\sqrt{a^2 - x^2}}\,dx &= \int \frac{1}{a\cos\theta}(a\cos\theta d\theta) \\ &= \int d\theta \\ &= \theta + C \\ &= \sin^{-1}\frac{x}{a} + C\end{aligned}$$

Proof

2. *Let* $x = a\tan\theta$. *Then*

$$\begin{aligned} a^2 + x^2 &= a^2 + a^2\tan^2\theta = a^2\sec^2\theta \\ dx &= a\sec^2\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{a^2+x^2}\,dx &= \int \frac{1}{a^2\sec^2\theta}(a\sec^2\theta d\theta) \\ &= \frac{1}{a}\int d\theta \\ &= \frac{\theta}{a} + C \\ &= \frac{1}{a}\tan^{-1}\frac{x}{a} + C \end{aligned}$$

Proof.

3. Let $x = a\sec\theta$. Then

$$\begin{aligned} x\sqrt{x^2-a^2} &= a\sec\theta\sqrt{a^2\sec^2\theta - a^2} = a^2\sec\theta\tan\theta \\ dx &= a\sec\theta\tan\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{x\sqrt{x^2-a^2}}\,dx &= \int \frac{1}{a^2\sec\theta\tan\theta}(a\sec\theta\tan\theta d\theta) \\ &= \frac{1}{a}\int d\theta \\ &= \frac{\theta}{a} + C \\ &= \frac{1}{a}\cos^{-1}\frac{a}{x} + C \end{aligned}$$

Note that $\theta = \cos^{-1}\frac{a}{x}$ since $\cos\theta = \frac{1}{\sec\theta} = \frac{a}{x}$.

□

Example

Use trigonometric substitution to evaluate the following integrals.

1. $\displaystyle\int \sqrt{1-x^2}\,dx$
2. $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx$
3. $\displaystyle\int \frac{x^3}{\sqrt{4-x^2}}\,dx$
4. $\displaystyle\int \frac{1}{(9+x^2)^2}\,dx$

Solution

1. *Let* $x = \sin\theta$. *Then*

$$\begin{aligned} \sqrt{1-x^2} &= \sqrt{1-\sin^2\theta} = \cos\theta \\ dx &= \cos\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \sqrt{1-x^2}\,dx &= \int \cos^2\theta d\theta \\ &= \int \frac{\cos 2\theta + 1}{2} d\theta \\ &= \frac{\sin 2\theta}{4} + \frac{\theta}{2} + C \\ &= \frac{\sin\theta\cos\theta}{2} + \frac{\sin^{-1}x}{2} + C \\ &= \frac{x\sqrt{1-x^2}}{2} + \frac{\sin^{-1}x}{2} + C \end{aligned}$$

Solution

2. *Let* $x = \tan\theta$. *Then*

$$\begin{aligned} 1 + x^2 &= 1 + \tan^2\theta = \sec^2\theta \\ dx &= \sec^2\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{\sqrt{1+x^2}}\,dx &= \int \frac{1}{\sec x}(\sec^2\theta d\theta) \\ &= \int \sec\theta d\theta \\ &= \ln|\tan\theta + \sec\theta| + C \\ &= \ln(x + \sqrt{1+x^2}) + C \end{aligned}$$

Solution

3. *Let* $x = 2\sin\theta$. *Then*

$$\begin{aligned}\sqrt{4-x^2} &= \sqrt{4-4\sin^2\theta} = 2\cos\theta \\ dx &= 2\cos\theta d\theta.\end{aligned}$$

Therefore

$$\begin{aligned}\int \frac{x^3}{\sqrt{4-x^2}}dx &= \int \frac{8\sin^3\theta}{2\cos\theta}(2\cos\theta d\theta) \\ &= 8\int \sin^3\theta d\theta \\ &= -8\int (1-\cos^2\theta) d\cos\theta \\ &= 8\left(\frac{\cos^3\theta}{3} - \cos\theta\right) + C \\ &= \frac{(4-x^2)^{\frac{3}{2}}}{3} - 4(4-x^2)^{\frac{1}{2}} + C\end{aligned}$$

Solution

4. *Let* $x = 3\tan\theta$. *Then*

$$\begin{aligned} 9 + x^2 &= 9 + 9\tan^2\theta = 9\sec^2\theta \\ dx &= 3\sec^2\theta d\theta. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{1}{(9+x^2)^2}\,dx &= \int \frac{1}{81\sec^4\theta}(3\sec^2\theta d\theta) = \frac{1}{27}\int \cos^2\theta d\theta \\ &= \frac{1}{54}\int (\cos 2\theta + 1) d\theta = \frac{1}{54}\left(\frac{\sin 2\theta}{2} + \theta\right) + C \\ &= \frac{1}{54}(\cos\theta\sin\theta + \theta) + C \\ &= \frac{1}{54}\left(\frac{3}{\sqrt{9+x^2}} \cdot \frac{x}{\sqrt{9+x^2}} + \tan^{-1}\frac{x}{3}\right) + C \\ &= \frac{x}{18(9+x^2)} + \frac{1}{54}\tan^{-1}\frac{x}{3} + C \end{aligned}$$

Definition (Rational functions)

A rational function is a function of the form

$$R(x) = \frac{f(x)}{g(x)}$$

where $f(x), g(x)$ are polynomials with real coefficients with $g(x) \neq 0$.

Techniques

We can integrate a rational function $R(x)$ with the following two steps.

1. *Find the partial fraction decomposition of $R(x)$, that is, expressing $R(x)$ in the form*

$$R(x) = q(x) + \sum \frac{A}{(x-\alpha)^k} + \sum \frac{B(x+a)}{((x+a)^2+b^2)^k} + \sum \frac{C}{((x+a)^2+b^2)^k}$$

   where $q(x)$ is a polynomial, A, B, C, α, a, b represent real numbers and k represents positive integer.

2. *Integrate the partial fraction.*

Theorem

Let $R(x) = \dfrac{f(x)}{g(x)}$ *be a rational function. We may assume that the leading coefficient of* $g(x)$ *is 1.*

1. *(Division algorithm for polynomials) There exists polynomials* $q(x), r(x)$ *with* $\deg(r(x)) < \deg(d(x))$ *or* $r(x) = 0$ *such that*

$$R(x) = q(x) + \frac{r(x)}{g(x)}.$$

   $q(x)$ *and* $r(x)$ *are the quotient and remainder of the division* $f(x)$ *by* $g(x)$.

2. *(Fundamental theorem of algebra for real polynomials)* $g(x)$ *can be written as a product of linear or quadratic polynomials. More precisely, there exists real numbers* $\alpha_1, \dots, \alpha_m, a_1, \dots, a_n, b_1, \dots, b_n$ *and positive integers* $k_1, \dots, k_m, l_1, \dots, l_n$ *such that*

$$g(x) = (x - \alpha_1)^{k_1} \cdots (x - \alpha_k)^{k_m}((x + a_1)^2 + b_1^2)^{l_1} \cdots ((x + a_n)^2 + b)_n^2)^{l_n}.$$

Techniques

Partial fractions can be integrated using the formulas below.

- $$\int \frac{dx}{(x-\alpha)^k} = \begin{cases} \ln|x-\alpha| + C, & \text{if } k = 1 \\ -\dfrac{1}{(k-1)(x-\alpha)^{k-1}} + C, & \text{if } k > 1 \end{cases}$$

- $$\int \frac{xdx}{(x^2+a^2)^k} = \begin{cases} \dfrac{1}{2}\ln(x^2+a^2) + C, & \text{if } k = 1 \\ -\dfrac{1}{2(k-1)(x^2+a^2)^{k-1}} + C, & \text{if } k > 1 \end{cases}$$

- $$\int \frac{dx}{(x^2+a^2)^k}$$
  $$= \begin{cases} \dfrac{1}{a}\tan^{-1}\dfrac{x}{a} + C, & \text{if } k = 1 \\ \dfrac{x}{2a^2(k-1)(x^2+a^2)^{k-1}} + \dfrac{2k-3}{2a^2(k-1)}\displaystyle\int \frac{dx}{(x^2+a^2)^{k-1}}, & \text{if } k > 1 \end{cases}$$

Theorem

Suppose $\dfrac{f(x)}{g(x)}$ *is a rational function such that the degree of* $f(x)$ *is smaller than the degree of* $g(x)$ *and* $g(x)$ *has only simple real roots, i.e.,*

$$g(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$$

for distinct real numbers $\alpha_1, \alpha_2, \cdots, \alpha_k$ *and* $a \neq 0$. *Then*

$$\frac{f(x)}{g(x)} = \frac{f(\alpha_1)}{g'(\alpha_1)(x-\alpha_1)} + \frac{f(\alpha_2)}{g'(\alpha_2)(x-\alpha_2)} + \cdots + \frac{f(\alpha_k)}{g'(\alpha_k)(x-\alpha_k)}$$

Proof

First, observe that

$$g'(x) = \sum_{j=1}^{k} a(x-\alpha_1)(x-\alpha_2)\cdots\widehat{(x-\alpha_j)}\cdots(x-\alpha_k)$$

where $\widehat{(x-\alpha_i)}$ *means the factor* $x-\alpha_i$ *is omitted. Thus we have*

$$\begin{aligned} g'(\alpha_i) &= \sum_{j=1}^{k} a(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots\widehat{(\alpha_i-\alpha_j)}\cdots(\alpha_i-\alpha_k) \\ &= a(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots\widehat{(\alpha_i-\alpha_i)}\cdots(\alpha_i-\alpha_k) \end{aligned}$$

Since $g(x)$ *has distinct real zeros, the partial fraction decomposition takes the form*

$$\frac{f(x)}{g(x)} = \frac{A_1}{x-\alpha_1} + \frac{A_2}{x-\alpha_2} + \cdots + \frac{A_k}{x-\alpha_k}.$$

Proof.

Multiplying both sides by $g(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_k)$, we get

$$f(x) = \sum_{i=1}^{k} A_i a(x-\alpha_1)(x-\alpha_2)\cdots\widehat{(x-\alpha_i)}\cdots(x-\alpha_k)$$

For $i = 1, 2, \cdots, k$, substituting $x = \alpha_i$, we obtain

$$\begin{aligned} f(\alpha_i) &= \sum_{j=1}^{k} A_j a(\alpha_j-\alpha_1)(\alpha_j-\alpha_2)\cdots\widehat{(\alpha_j-\alpha_i)}\cdots(\alpha_j-\alpha_k) \\ &= A_i a(\alpha_i-\alpha_1)(\alpha_i-\alpha_2)\cdots\widehat{(\alpha_i-\alpha_i)}\cdots(\alpha_i-\alpha_k) \\ &= A_i g'(\alpha_i) \end{aligned}$$

and the result follows. □

Example

Evaluate the following integrals.

1. $$\int \frac{x^5 + 2x - 1}{x^3 - x}\, dx$$
2. $$\int \frac{9x - 2}{2x^3 + 3x^2 - 2x}\, dx$$
3. $$\int \frac{x^2 - 2}{x(x-1)^2}\, dx$$
4. $$\int \frac{x^2}{x^4 - 1}\, dx$$
5. $$\int \frac{8x^2}{x^4 + 4}\, dx$$
6. $$\int \frac{2x + 1}{x^4 + 2x^2 + 1}\, dx$$

Solution

1. *By division and factorization* $x^3 - x = x(x-1)(x+1)$, *we obtain the partial fraction decomposition*

$$\frac{x^5+4x-3}{x^3-x} = x^2+1+\frac{5x-3}{x^3-x} = x^2+1+\frac{A}{x}+\frac{B}{x-1}+\frac{C}{x+1}.$$

Multiply both sides by $x(x-1)(x+1)$ *and obtain*

$$\begin{aligned} & 5x-3 = A(x-1)(x+1)+Bx(x+1)+Cx(x-1) \\ \Rightarrow \quad & A=3, B=1, C=-4. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{x^5+4x-3}{x^3-x}\,dx &= \int \left(x^2+1+\frac{3}{x}+\frac{1}{x-1}-\frac{4}{x+1}\right)dx \\ &= \frac{x^3}{3}+x+3\ln|x|+\ln|x-1|-4\ln|x+1|+C. \end{aligned}$$

Solution

2. *By factorization* $2x^3 + 3x^2 - 2x = x(x+2)(2x-1)$, *we obtain the partial fraction decomposition*

$$\frac{9x-2}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{2x-1}.$$

Multiply both sides by $x(x+2)(2x-1)$ *and obtain*

$$\begin{aligned} & 9x - 2 = A(x+2)(2x-1) + Bx(2x-1) + Cx(x+2) \\ \Rightarrow \quad & A = 1, B = -2, C = 2. \end{aligned}$$

Therefore

$$\begin{aligned} & \int \frac{9x-2}{2x^3+3x^2-2x}\,dx \\ = \quad & \int \left(\frac{1}{x} - \frac{2}{x+2} + \frac{2}{2x-1}\right) dx \\ = \quad & \ln|x| - 2\ln|x+2| + \ln|2x-1| + C. \end{aligned}$$

Solution

3. *The partial fraction decomposition is*

$$\frac{x^2-2}{x(x-1)^2} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x}.$$

Multiply both sides by $x(x-1)^2$ *and obtain*

$$\begin{aligned} & x^2-2 = Ax + Bx(x-1) + C(x-1)^2 \\ \Rightarrow \quad & A=-1, B=3, C=-2. \end{aligned}$$

Therefore

$$\begin{aligned} \int \frac{x^2-2}{x(x-1)^2}\,dx &= \int \left(-\frac{1}{(x-1)^2} + \frac{3}{x-1} - \frac{2}{x}\right) dx \\ &= \frac{1}{x-1} + 3\ln|x-1| - 2\ln|x| + C. \end{aligned}$$

Solution

4. *The partial fraction decomposition is*

$$\begin{aligned}\frac{x^2}{x^4-1} &= \frac{x^2}{(x^2-1)(x^2+1)}\\ &= \frac{1}{2}\left(\frac{1}{x^2-1}+\frac{1}{x^2+1}\right)\\ &= \frac{1}{2(x-1)(x+1)}+\frac{1}{2(x^2+1)}\\ &= \frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{2(x^2+1)}\end{aligned}$$

Therefore

$$\begin{aligned}\int\frac{x^2dx}{x^4-1} &= \int\left(\frac{1}{4(x-1)}-\frac{1}{4(x+1)}+\frac{1}{2(x^2+1)}\right)dx\\ &= \frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|+\frac{1}{2}\tan^{-1}x+C\end{aligned}$$

Solution

5. *By factorization* $x^4 + 4 = (x^2+2)^2 - (2x)^2 = (x^2-2x+2)(x^2+2x+2)$,

$$\begin{aligned}
& \int \frac{8x^2}{x^4+4}\,dx \\
= & \int \frac{8x^2dx}{(x^2-2x+2)(x^2+2x+2)}\,dx \\
= & \int 2x\left(\frac{4x}{(x^2-2x+2)(x^2+2x+2)}\right)dx \\
= & \int 2x\left(\frac{1}{x^2-2x+2} - \frac{1}{x^2+2x+2}\right)dx \\
= & \int \left(\frac{2x}{(x-1)^2+1} - \frac{2x}{(x+1)^2+1}\right)dx \\
= & \int \left(\frac{2(x-1)}{(x-1)^2+1} + \frac{2}{(x-1)^2+1} - \frac{2(x+1)}{(x+1)^2+1} + \frac{2}{(x+1)^2+1}\right)dx \\
= & \ln(x^2-2x+2) + 2\tan^{-1}(x-1) - \ln(x^2+2x+2) + 2\tan^{-1}(x+1) + C
\end{aligned}$$

Solution

6.
$$\begin{aligned}
&\int \frac{2x+1}{x^4+2x^2+1}\,dx\\
=&\int \frac{2xdx}{(x^2+1)^2}+\int \frac{dx}{(x^2+1)^2}\\
=&\int \frac{d(x^2+1)}{(x^2+1)^2}+\int \frac{x^2+1}{(x^2+1)^2}\,dx-\int \frac{x^2dx}{(x^2+1)^2}\\
=&-\frac{1}{x^2+1}+\int \frac{dx}{x^2+1}-\frac{1}{2}\int \frac{xd(x^2+1)}{(x^2+1)^2}\\
=&-\frac{1}{x^2+1}+\tan^{-1}x+\frac{1}{2}\int xd\left(\frac{1}{x^2+1}\right)\\
=&-\frac{1}{x^2+1}+\tan^{-1}x+\frac{1}{2}\left(\frac{x}{x^2+1}\right)-\frac{1}{2}\int \frac{dx}{x^2+1}\\
=&\frac{x-2}{2(x^2+1)}+\frac{1}{2}\tan^{-1}x+C
\end{aligned}$$

Example

Find the partial fraction decomposition of the following functions.

1. $\dfrac{5x-3}{x^3-x}$
2. $\dfrac{9x-2}{2x^3+3x^2-2x}$

Solution

1. *For* $g(x) = x^3 - x = x(x-1)(x+1)$, $g'(x) = 3x^2 - 1$. *Therefore*

$$\begin{aligned}\frac{5x-3}{x^3-x} &= \frac{-3}{g'(0)x} + \frac{5(1)-3}{g'(1)(x-1)} + \frac{5(-1)-3}{g'(-1)(x+1)} \\ &= \frac{3}{x} + \frac{1}{x-1} - \frac{4}{x+1}\end{aligned}$$

2. *For* $g(x) = 2x^3 + 3x^2 - 2x = x(x+2)(2x-1)$, $g'(x) = 6x^2 + 6x - 2$. *Therefore*

$$\begin{aligned}&\frac{9x-2}{2x^3+3x^2-2x} \\ &= \frac{-2}{g'(0)x} + \frac{9(-2)-2}{g'(-2)(x+2)} + \frac{9(\frac{1}{2})-2}{g'(\frac{1}{2})(2x-1)} \\ &= \frac{1}{x} - \frac{2}{x+2} + \frac{2}{2x-1}\end{aligned}$$

□

Techniques

To evaluate

$$\int R(\cos x, \sin x, \tan x)dx$$

where R is a rational function, we may use t-substitution

$$t = \tan\frac{x}{2}.$$

Then

$$\tan x = \frac{2t}{1-t^2};\ \cos x = \frac{1-t^2}{1+t^2};\ \sin x = \frac{2t}{1+t^2};$$

$$dx = d(2\tan^{-1} t) = \frac{2dt}{1+t^2}.$$

We have

$$\int R(\cos x, \sin x, \tan x)dx = \int R\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, \frac{2t}{1-t^2}\right)\frac{2dt}{1+t^2}$$

which is an integral of rational function.

Example

Use t-substitution to evaluate the following integrals.

1. $$\int \frac{dx}{1+\cos x}$$
2. $$\int \frac{\sin x dx}{\cos x + \sin x}$$
3. $$\int \frac{dx}{1+\cos x + \sin x}$$

Solution

1. *Let* $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$. *We have*

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \left(\frac{1}{1+\frac{1-t^2}{1+t^2}}\right)\frac{2dt}{1+t^2} = \int dt = t + C = \tan\frac{x}{2} + C \\ &= \frac{\sin\frac{x}{2}}{\cos\frac{x}{2}} + C = \frac{2\cos\frac{x}{2}\sin\frac{x}{2}}{2\cos^2\frac{x}{2}} + C = \frac{\sin x}{1+\cos x} + C\end{aligned}$$

Alternatively

$$\begin{aligned}\int \frac{dx}{1+\cos x} &= \int \frac{dx}{2\cos^2\frac{x}{2}} = \frac{1}{2}\int \sec^2\frac{x}{2}\,dx \\ &= \tan^{-1}\frac{x}{2} + C = \frac{\sin x}{1+\cos x} + C\end{aligned}$$

Solution

2. *Let* $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$. *We have*

$$\begin{aligned}
\int \frac{\sin x dx}{\cos x + \sin x} &= \int \frac{\frac{2t}{1+t^2}}{\frac{1-t^2}{1+t^2} + \frac{2t}{1+t^2}} \frac{2dt}{1+t^2} \\
&= \int \left(\frac{1}{1+t^2} + \frac{t}{1+t^2} + \frac{t-1}{1+2t-t^2} \right) dt \\
&= \tan^{-1} t - \frac{1}{2} \ln \left| \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} \right| + C \\
&= \frac{x}{2} - \frac{1}{2} \ln|\cos x + \sin x| + C
\end{aligned}$$

Alternatively

$$\begin{aligned}
\int \frac{\sin x dx}{\cos x + \sin x} &= \frac{1}{2} \int \left(1 - \frac{\cos x - \sin x}{\cos x + \sin x} \right) dx \\
&= \frac{x}{2} - \frac{1}{2} \int \frac{d(\sin x + \cos x)}{\cos x + \sin x} = \frac{x}{2} - \frac{1}{2} \ln|\cos x + \sin x| + C
\end{aligned}$$

Solution

3. *Let* $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$. *We have*

$$\begin{aligned}
\int \frac{dx}{1+\cos x+\sin x} &= \int \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \\
&= \int \frac{dt}{1+t} \\
&= \ln|1+t| + C \\
&= \ln\left|1+\tan\frac{x}{2}\right| + C \\
&= \ln\left|1+\frac{\sin x}{1+\cos x}\right| + C \\
&= \ln\left|\frac{1+\cos x+\sin x}{1+\cos x}\right| + C
\end{aligned}$$