THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 8 Solutions

Problems that may be demonstrated in class :

Q1. Evaluate $\int f(x)dx$ for different functions $f(x)$ as below:

(a) $x\sqrt{x-4}$; (b) $\frac{2x-5}{x^2-5x+36}$; (c) $3^{e^x+1}e^x$; (d) $\sin^2 x$;

(e) $\sin^3 x$; (f) $\tan x$; (g) $\tan^2 x$; (h) $\tan^3 x$;

(i) $\sec x$; (j) $\frac{1}{\sqrt{x^2+4}}$; (k) $\sin 7x\cos 4x$; (l) $\frac{\arctan x}{x^2+1}$.

Q2. In this question, we study the behaviour of the indefinite integral $\int \frac{P(x)}{ax^2+bx+c}dx$, where $P(x)$ is a polynomial and $a, b, c \in \mathbb{R}$ with $a \neq 0$.

(a) Find the discriminant of the equation $x^2 - 5x + 6 = 0$. Find real constants A and B such that $\frac{x-5}{x^2-5x+6} \equiv \frac{A}{x-2} + \frac{B}{x-3}$. Hence evaluate $\int \frac{x-5}{x^2-5x+6}dx$.

(b) Find the discriminant of the equation $x^2 + 2x + 1 = 0$. Find real constants A and B such that $\frac{3x+2}{x^2+2x+1} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2}$. Hence evaluate $\int \frac{3x+2}{x^2+2x+1}dx$.

(c) Find the discriminant of the equation $x^2 + 2x + 2 = 0$. Evaluate $\int \frac{2x+3}{x^2+2x+2}dx$.

(d) Find real constants A, B, C, D such that $\frac{x^3-3x^2-3x+7}{x^2-5x+6} \equiv Ax + B + \frac{Cx+D}{x^2-5x+6}$. Hence evaluate $\int \frac{x^3-3x^2-3x+7}{x^2-5x+6}dx$.

Q3. Let $t = \tan\frac{\theta}{2}$. Express $\sin\theta$ and $\cos\theta$ in terms of t. Evaluate $\int \frac{d\theta}{\sin\theta - \cos\theta - 1}$.

Solutions :

Q1. (a) Let $u = x - 4$. Then $du = dx$.

$$\int x\sqrt{x-4}dx = \int (u+4)\sqrt{u}du = \int (u^{\frac{3}{2}} + 4u^{\frac{1}{2}})du = \frac{2}{5}u^{\frac{5}{2}} + \frac{8}{3}u^{\frac{3}{2}} + C$$
$$= \frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + C.$$

(b) Let $u = x^2 - 5x + 36$. Then $du = (2x-5)dx$.

$$\int \frac{2x-5}{x^2-5x+36}dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2 - 5x + 36| + C.$$

(c) Let $u = e^x$. Then $du = e^x dx$.

$$\int 3^{e^x+1}e^x dx = \int 3^{u+1}du = 3\int 3^u du = \frac{3^{u+1}}{\ln 3} + C = \frac{3^{e^x+1}}{\ln 3} + C.$$

(d)

$$\int \sin^2 x dx = \int \frac{1}{2}(1 - \cos 2x)dx = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

(e) Let $u = \cos x$. Then $du = -\sin x dx$.

$$\int \sin^3 x dx = \int \sin x(1-\cos^2 x)dx = \int \sin x dx - \int \sin x \cos^2 x dx$$
$$= -\cos x + \int u^2 du = -\cos x + \frac{u^3}{3} + C = -\cos x + \frac{\cos^3 x}{3} + C.$$

(f) Let $u = \cos x$. Then $du = -\sin x dx$.

$$\int \tan x dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{du}{u} = -\ln|u| + C = -\ln|\cos x| + C.$$

(g)

$$\int \tan^2 x dx = \int (\sec^2 x - 1)dx = \tan x - x + C.$$

(h) Let $u = \tan x$. Then $du = \sec^2 x dx$.

$$\int \tan^3 x dx = \int \tan x(\sec^2 x - 1)dx = \int \tan x \sec^2 x dx - \int \tan x dx$$
$$= \int u du - \ln|\cos x| = \frac{u^2}{2} - \ln|\cos x| + C = \frac{\tan^2 x}{2} - \ln|\cos x| + C.$$

(i) Let $u = \tan x + \sec x$. Then $du = (\sec^2 x + \tan x \sec x)dx = \sec x(\tan x + \sec x)dx$.

$$\int \sec x dx = \int \frac{\sec x(\tan x + \sec x)}{\tan x + \sec x} dx = \int \frac{du}{u} = \ln|u| + C$$
$$= \ln|\tan x + \sec x| + C.$$

(j) Let $x = 2\tan\theta$, $-\pi/2 < \theta < \pi/2$. Then $dx = 2\sec^2\theta d\theta$.

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta d\theta}{\sqrt{4\tan^2\theta + 4}} = \int \frac{\sec^2\theta d\theta}{\sqrt{\sec^2\theta}} = \int \sec\theta d\theta$$
$$= \ln|\tan\theta + \sec\theta| + C' = \ln\left|\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + 1}\right| + C'$$
$$= \ln\left|\frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + 1}\right| + \ln 2 + C = \ln|x + \sqrt{x^2+4}| + C.$$

(k) $\sin 7x \cos 4x = \frac{1}{2}(\sin(7+4)x + \sin(7-4)x) = \frac{1}{2}(\sin 11x + \sin 3x)$. Thus

$$\int \sin 7x \cos 4x dx = \frac{1}{2}\int (\sin 11x + \sin 3x)dx = -\frac{\cos 11x}{22} - \frac{\cos 3x}{6} + C.$$

(l) Let $u = \arctan x$. Then $du = \frac{dx}{x^2+1}$.

$$\int \frac{\arctan x}{x^2+1} dx = \int u du = \frac{u^2}{2} + C = \frac{(\arctan x)^2}{2} + C.$$

Q2. (a) The discriminant is $(-5)^2 - 4(1)(6) = 1 > 0$.

$$\frac{x-5}{(x-2)(x-3)} \equiv \frac{x-5}{x^2-5x+6} \equiv \frac{A}{x-2} + \frac{B}{x-3},$$
$$x - 5 \equiv A(x-3) + B(x-2).$$

Putting $x = 2$, we get $A = \frac{2-5}{2-3} = 3$; putting $x = 3$, we get $B = \frac{3-5}{3-2} = -2$.

$$\int \frac{x-5}{x^2-5x+6} dx = \int \left(\frac{3}{x-2} - \frac{2}{x-3} \right) dx = 3 \ln|x-2| - 2 \ln|x-3| + C.$$

(b) The discriminant is $2^2 - 4(1)(1) = 0$.

$$\frac{3x+2}{(x+1)^2} \equiv \frac{3x+2}{x^2+2x+1} \equiv \frac{A}{x+1} + \frac{B}{(x+1)^2},$$
$$3x + 2 \equiv A(x+1) + B \equiv Ax + A + B.$$

Comparing coefficients of x, $A = 3$. Putting $x = -1$, $B = 3(-1) + 2 = -1$.

$$\int \frac{3x+2}{(x+1)^2} dx = \int \left(\frac{3}{x+1} - \frac{1}{(x+1)^2} \right) dx = 3 \ln|x+1| + \frac{1}{x+1} + C.$$

(c) The discriminant is $2^2 - 4(1)(2) = -4 < 0$.

$$\int \frac{2x+3}{x^2+2x+2} dx = \int \frac{2x+2}{x^2+2x+2} dx + \int \frac{dx}{(x+1)^2+1}.$$

Let $u = x^2 + 2x + 2$. Then $du = (2x+2)dx$.

$$\int \frac{2x+2}{x^2+2x+2} dx = \int \frac{du}{u} = \ln|u| + C = \ln|x^2+2x+2| + C.$$

Let $x = \tan\theta - 1$, $-\pi/2 < \theta < \pi/2$. Then $dx = \sec^2\theta d\theta$.

$$\int \frac{dx}{(x+1)^2+1} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta + 1} = \int d\theta = \theta = \arctan(x+1) + C.$$

Therefore,

$$\int \frac{2x+3}{x^2+2x+2} dx = \ln|x^2+2x+2| + \arctan(x+1) + C.$$

(d) We can perform long division.

$$\begin{array}{r|l} & \quad\quad\quad\quad\;\; x \;\; +2 \\ \hline x^2-5x+6 & x^3 - 3x^2 - 3x + 7 \\ & -x^3 + 5x^2 - 6x \\ \hline & \quad\;\; 2x^2 - 9x + 7 \\ & \;\; -2x^2 + 10x - 12 \\ \hline & \quad\quad\quad\quad\; x - 5 \end{array}$$

Then

$$\frac{x^3-3x^2-3x+7}{x^2-5x+6} \equiv x + 2 + \frac{x-5}{x^2-5x+6}.$$

Therefore, $A = 1$, $B = 2$, $C = 1$ and $D = -5$. By part (a),

$$\begin{aligned}\int \frac{x^3 - 3x^2 - 3x + 7}{x^2 - 5x + 6} dx &= \int \left(x + 2 + \frac{x - 5}{x^2 - 5x + 6} \right) dx \\ &= \int (x + 2) dx + \int \frac{x - 5}{x^2 - 5x + 6} dx \\ &= \frac{x^2}{2} + 2x + 3 \ln|x - 2| - 2 \ln|x - 3| + C.\end{aligned}$$

Q3.

$$\sin\theta = 2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2} = \frac{2\tan\frac{\theta}{2}}{\sec^2\frac{\theta}{2}} = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2t}{1+t^2}.$$

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{\sec^2\frac{\theta}{2}} - 1 = \frac{2}{1+\tan^2\frac{\theta}{2}} - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}.$$

$dt = \frac{1}{2}\sec^2\frac{\theta}{2} d\theta = \frac{1+t^2}{2} d\theta$. Hence $d\theta = \frac{2dt}{1+t^2}$.

$$\begin{aligned}\int \frac{d\theta}{\sin\theta - \cos\theta - 1} &= \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2} - \frac{1-t^2}{1+t^2} - 1} = \int \frac{2dt}{2t - (1-t^2) - (1+t^2)} \\ &= \int \frac{dt}{t-1} = \ln|t-1| + C = \ln|\tan\tfrac{\theta}{2} - 1| + C.\end{aligned}$$