THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 7 Solutions

Problems that may be demonstrated in class :

Q1. Consider $f(x) = -\ln(1-x)$. Compute its Taylor series at $x=0$ and show that if $p_4(x)$ is the Taylor polynomial of degree 4 of f at $x=0$, then

$$|f(0.1) - p_4(0.1)| < 10^{-5}.$$

Hence compute the value of $\ln 0.9$ by hand up to 4 decimal place.

Q2. Given that $\frac{1}{x^2-3x+2} = \frac{1}{x-2} - \frac{1}{x-1}$, find the Taylor series of $\frac{1}{x^2-3x+2}$ at $x=0$.

Q3. Find the Taylor series of $\frac{1}{1-x}$ at $x=2$.

Q4. Let f be a differentiable function satisfying $f'(x) = 1-x+f(x)$ for all $x \in \mathbb{R}$ and $f(0) = 2$. Show that $f(x)$ is infinitely differentiable and find its Taylor series at $x=0$.

Q5. Let $f(x) = \begin{cases} \frac{\sin x - x}{x^3} & \text{if } x \neq 0 \\ -\frac{1}{6} & \text{if } x = 0 \end{cases}$. Use Taylor theorem to show that $-\frac{1}{6} \leq f(x) \leq -\frac{1}{6} + \frac{x^2}{120}$ for $x \in \mathbb{R}$.

Solutions:

Notice that if $g_1(x) = (a-x)^{-k}$ where k is a positive integer and $a \in \mathbb{R}$, then $g_1^{(n)} = \frac{(k+n-1)!}{(k-1)!}(a-x)^{-k-n}$ for all positive integer n. Also if $g_2(x) = h(-x)$ for some infinitely differentiable function h, then $g_2^{(n)}(x) = (-1)^n h^{(n)}(-x)$ for all positive integer n.

Q1. Since $f^{(n)}(x) = (n-1)!(1-x)^{-n}$ for all positive integer n, $f^n(0) = (n-1)!$ and $f(0) = -\ln 1 = 0$ and the Taylor series for f at $x=0$ is

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=1}^{\infty} \frac{(k-1)!}{k!}x^k = \sum_{k=1}^{\infty} \frac{x^k}{k} = x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots.$$

So by Taylor theorem, there exists $\xi \in (0, 0.1)$ such that $|f(0.1)-p_4(0.1)| = |\frac{f^{(5)}(\xi)}{5!}0.1^5| = \frac{|1-\xi|^{-5}}{5}10^{-5} \leq \frac{0.9^{-5}}{5}10^{-5} < 10^{-5}$. Now we compute $p_4(0.1) = 0.1 + 0.1^2/2 + 0.1^3/3 + 0.1^4/4 = 0.1+0.005+0.000333\ldots+0.000025 = 0.105358333\ldots$. So $|\ln 0.9 + 0.10535833333| \leq |-f(0.1) + p_4(0.1)| + 10^{-8}/3 < 2 \cdot 10^{-5}$ and $\ln 0.9 = -0.1054$ correct to 4 decimal place.

Q2. Let $f_a(x) = \frac{1}{x-a}$ for $a > 0$. $f^{(n)}(x) = (-1)^n n!(x-a)^{-n-1}$ and $f^{(n)}(0) = (-1)^n n!(-a)^{-n-1} = -n!a^{-n-1}$. The Taylor series of $\frac{1}{x-2}$ at $x=0$ is $\displaystyle\sum_{k=0}^{\infty} \frac{f_2^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} -2^{-k-1}x^k$ and the Taylor series of $\frac{1}{x-2}$ at $x=0$ is $\displaystyle\sum_{k=0}^{\infty} \frac{f_1^{(k)}(0)}{k!}x^k = \sum_{k=0}^{\infty} -1^{-k-1}x^k = \sum_{k=0}^{\infty} -x^k$. Therefore the Taylor series of $\frac{1}{x^2-3x+2}$ is $\displaystyle\sum_{k=0}^{\infty} -2^{-k-1}x^k - \sum_{k=0}^{\infty} -x^k = \sum_{k=0}^{\infty}(-2^{-k-1}+1)x^k$.

Q3. Let $f(x) = (1-x)^{-1}$, then $f^{(n)}(x) = n!(1-x)^{-n-1}$ and $f^{(n)}(2) = n!(1-2)^{-n-1} = (-1)^{n+1}n!$ for all positive integer n. Therefore the Taylor series at $x = 2$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \sum (-1)^{n+1} x^n.$$

Q4. Since $f'(x) = 1 - x + f(x)$ which is a sum of differentiable functions, $f'(x)$ is differentiable and $f''(x) = -1 + f'(x)$. This again is differentiable for the same reason. $f'''(x) = f''(x)$ and so $f^{(n+1)}(x) = f^{(n)}(x)$ for $n \geq 2$. Therefore f is infinitely differentiable. Now $f(0) = 2$ so $f'(0) = 1 - 0 + f(0) = 3$, $f''(0) = -1 + f'(0) = 2$ and $f^{(n)}(0) = f^{(2)}(0) = 2$ for $n \geq 3$. So the Taylor series of $f(x)$ at $x = 0$ is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = 2 + 3x + \sum_{n=2}^{\infty} \frac{2}{n!} x^n.$$

Q5. The Taylor series of $\sin x$ at $x = 0$ is $x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots$. For $x \neq 0$, by Taylor theorem, $\sin x - (x - \frac{x^3}{6}) = \frac{\sin(\xi)}{4!} x^4$ and $\sin x - (x - \frac{x^3}{6} + \frac{x^5}{120}) = \frac{-\sin(\zeta)}{6!} x^6$ for some ξ and ζ between 0 and x. So $\frac{\sin x - x}{x^3} + \frac{1}{6} = \frac{\sin(\xi)}{4!} x$ and $\frac{\sin x - x}{x^3} + \frac{1}{6} - \frac{x^2}{120} = \frac{-\sin(\zeta)}{6!} x^3$. For $x \in [-\pi, \pi]$, $\sin(\xi)x \geq 0$ and $-\sin(\zeta)x^3 \leq 0$. Therefore $-\frac{1}{6} + \frac{x^2}{120} \geq f(x) \geq -\frac{1}{6}$ on $[-\pi, \pi]$.
For $x \geq \pi$, we use the following technique:

Lemma Let f, g be a continuously differentiable function on $\mathbb{R}$ and $a \in \mathbb{R}$, assume we have

1. $f(a) > g(a)$, and
2. $f'(x) > g'(x)$ for all $x > a$.

Then $f(x) > g(x)$ for all $x \geq a$.

Proof: Observe that $f - g$ is an increasing function on $x \geq a$.

Corollary Let f, g be n-times continuously differentiable function on $\mathbb{R}$ and $a \in \mathbb{R}$, assume we have

1. $f^{(k)}(a) > g^{(k)}(a)$ for $k = 0, 1, \ldots n-1$, and
2. $f^{(n)}(x) > g^{(n)}(x)$ for all $x > a$.

Then $f(x) > g(x)$ for all $x \geq a$.

Proof: Induction or by using Taylor theorem on $f - g$ at $x = a$.

Now $f'(x) = \frac{2x - 3\sin x + x\cos x}{x^4}$ for $x \neq 0$. Since $2x - 3\sin x + x\cos x = x(1 + \cos x) + (x - 3\sin x) > 0$ for $x \geq \pi$ and $2x - 3\sin x + x\cos x = x(1+\cos x) + (x - 3\sin x) < 0$ for $x \leq -\pi$, f is increasing on $x > \pi$ and decreasing on $x < -\pi$. So we have $f(x) \geq -\frac{1}{6}$ for all x. For the other inequality, let $g(x) = -\frac{1}{6} + \frac{x^2}{120}$. Then $g(\pi) > f(\pi)$ by previous step. To show $g'(x) > f'(x)$ for $x \geq \pi$, it suffices to show $x^5 > 60(2x - 3\sin x + x\cos x)$ for $x \geq \pi$. The case $x = \pi$ is just direct check. Notice that by differentiate once more, we see that we reduce the problem to showing $5x^4 - 120 + 180\cos x - 60\cos x + 60x\sin x > 0$ for $x \geq pi$. Observe that $5x^4 - 120 + 180\cos x - 60\cos x + 60x\sin x \geq 5x^4 - 240 - 60x > 0$ for $x \geq \pi$. Therefore by the lemma and corollary we have proven the inequalities for $x \geq \pi$. Finally, for $x \leq -\pi$, we observe that $f(x)$ and $-\frac{1}{6} + \frac{x^2}{120}$ are even functions.