THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 2

Definition An infinite sequence $\{a_n\}$ of real numbers is said to

- *converge* if there exists real number L s.t. for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ s.t. for any $n > N$, $|a_n - L| < \varepsilon$. In this cases, we write $\lim_{n\to\infty} a_n = L$.
- *diverge* if it does not converge.
- *tend to* $+\infty$ $(-\infty)$ if for any real number M, there exists $N \in \mathbb{N}$ s.t. for any $n > N$, $a_n > M$ $(a_n < M)$. In this case, we write $\lim_{n\to\infty} a_n = +\infty$ $(\lim_{n\to\infty} a_n = -\infty)$.
- be *monotonic increasing* (*decreasing*) if for any $m < n$, $a_m \le a_n$ $(a_m \ge a_n)$.
- be *strictly increasing* (*decreasing*) if for any $m < n$, $a_m < a_n$ $(a_m > a_n)$.
- be *bounded above* (*below*) if there exists real number M s.t. for any $n \in \mathbb{N}$, $a_n \le M$ $(a_n \ge M)$.
- be *bounded* if there exists real number M s.t. for any $n \in \mathbb{N}$, $|a_n| \le M$.

Theorems From now onwards, by a sequence we mean an infinite sequence of real numbers. Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences.

- If $\{a_n\}$ converges, then it is bounded.
- If $\{a_n\}$ is monotonic increasing and bounded above, then it converges.
- If $\{a_n\}$ is monotonic increasing and not bounded above, then it tends to $+\infty$.
- If $\{a_n\}$ is monotonic decreasing and bounded below, then it converges.
- If $\{a_n\}$ is monotonic decreasing and not bounded below, then it tends to $-\infty$.
- If $\{a_n\}$ and $\{b_n\}$ converge with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, then the sequences $\{a_n + b_n\}$, $\{a_nb_n\}$ and $\{|a_n|\}$ converge and

$$\lim_{n\to\infty}(a_n + b_n) = a + b, \quad \lim_{n\to\infty} a_nb_n = ab \quad \text{and} \quad \lim_{n\to\infty}|a_n| = |a|.$$

- If $\{a_n\}$ converges with $\lim_{n\to\infty} a_n = a \neq 0$, then $\{1/a_n\}$ converges and

$$\lim_{n\to\infty} 1/a_n = 1/a.$$

- If $\{|a_n|\}$ converges with $\lim_{n\to\infty}|a_n| = 0$, then $\{a_n\}$ converges and $\lim_{n\to\infty} a_n = 0$.
- If $\lim_{n\to\infty}|a_n| = +\infty$, then $\{1/a_n\}$ converges and $\lim_{n\to\infty} \frac{1}{a_n} = 0$.
- (Sandwich Theorem) if $a_n \le b_n \le c_n$ for any $n \in \mathbb{N}$ and $\{a_n\}$ and $\{c_n\}$ converge with $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$.
- If $a_n \le b_n$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = +\infty$, then $\lim_{n\to\infty} b_n = +\infty$.
- If $a_n \ge b_n$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = -\infty$, then $\lim_{n\to\infty} b_n = -\infty$.
- If $\{a_n\}$ converges with $\lim_{n\to\infty} a_n = 0$ and $\{b_n\}$ is bounded, then $\{a_nb_n\}$ converges and $\lim_{n\to\infty} a_nb_n = 0$.

- If $\lim_{n\to\infty} a_n = L$ (L can be any real number, $+\infty$ or $-\infty$), then for any subsequence $\{a_{n_k}\}$ of $\{a_n\}$, $\lim_{k\to\infty} a_{n_k} = L$.
- If $\lim_{n\to\infty} a_{2n-1} = \lim_{n\to\infty} a_{2n} = L$ (L can be any real number, $+\infty$ or $-\infty$), then $\lim_{n\to\infty} a_n = L$.
- Suppose $a \geq 0$. Then

$$\lim_{n\to\infty} a^n = \begin{cases} +\infty, & \text{if } a > 1; \\ 1, & \text{if } a = 1; \\ 0, & \text{if } 0 \leq a < 1. \end{cases}$$

- Let $P(x)$ and $Q(x)$ be polynomial functions with leading coefficients a and b respectively. Suppose $Q(x) \not\equiv 0$. Then

$$\lim_{n\to\infty} \frac{P(n)}{Q(n)} = \begin{cases} +\infty, & \text{if } \deg P > \deg Q \text{ and } ab > 0; \\ -\infty, & \text{if } \deg P > \deg Q \text{ and } ab < 0; \\ \frac{a}{b}, & \text{if } \deg P = \deg Q; \\ 0, & \text{if } \deg P < \deg Q. \end{cases}$$

Problems that may be demonstrated in class :

Assume we know the fact: $2 < e = \lim_{n\to\infty}(1+\frac{1}{n})^n$, $\lim_{n\to\infty} \sin\frac{1}{n} = 0$.

Q1. State whether the following sequence converges. Find the limit if it exists.

(a) $\frac{37(-n)^{2017}-(-n)^{689}}{141(-n)^{2017}+928(-n)^{64}}$; (b) $\sqrt[3]{2n^3+1} - \sqrt[3]{2n^3-n^2}$; (c) $(-1/2)^n$;

(d) $(1-\frac{1}{n+1})^n$; (e) $\sin\frac{n^2}{n+2} - \sin\frac{n^3-n-2}{n^2+2n}$; (f) $\frac{n^2}{\ln(n+1)}$;

(g) $\cos\frac{1}{n}$; (h) $\tan\frac{1}{n}$.

Q2. Let $\{a_n\}$ be a *harmonic sequence*, i.e. a sequence such that $a_n \neq 0$ for any $n \in \mathbb{N}$ and $1/a_n$ is an arithmetic sequence. Prove that $\{a_n\}$ converges.

Q3. Let $\{a_n\}$ be a sequence such that $a_n > 0$ for any $n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = a > 0$. Use Sandwich Theorem to show that $\{\sqrt{a_n}\}$ converges and $\lim_{n\to\infty}\sqrt{a_n} = \sqrt{a}$.

Q4. Suppose $\{a_n\}$ is a sequence such that $a_1 \neq 0$ and $a_{n+1} = 2^{-1}(a_n + a_n^{-1})$ for any $n \in \mathbb{N}$. Does $\{a_n\}$ converge? If it does, find its limit.

Q5. Suppose for any $m \in \mathbb{N}$, we have a function $f_m(x) = x^2 - mx - 1, x \in \mathbb{R}$ and a sequence $\{a_{m,n}\}$ satisfying the recursive relation:

$$a_{m,n+1} = m + \frac{1}{a_{m,n}} \quad \text{for any } n \in \mathbb{N}, \quad a_{m,1} > 0.$$

(a) Fix $m \in \mathbb{N}$. Show that for any $n \in \mathbb{N}$, $a_{m,n} > 0$ and

$$f_m(a_{m,n+1}) = -\frac{f_m(a_{m,n})}{a_{m,n}^2} = \frac{a_{m,n+1} - a_{m,n}}{a_{m,n}}.$$

(b) Fix $m \in \mathbb{N}$. Show that $\{a_{m,2n-1}\}$ is monotonic decreasing and bounded below if $f_m(a_{m,1}) \geq 0$ and $\{a_{m,2n-1}\}$ is a monotonic increasing and bounded above if $f_m(a_{m,1}) < 0$.

(c) Fix $m \in \mathbb{N}$. Show that $\{a_{m,n}\}$ converges and find its limit a_m in terms of m.

(d) Evaluate $\lim_{m\to\infty} a_m$ and $\lim_{m\to\infty}(a_{m+1} - a_m)$.