THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 11

More Techniques of Integration :

Trigonometric substitution: when the following expressions are encountered in the integrand of an integral, we may use trigonometric substitution:

Expression	Substitution	dx	$\sin\theta$	$\cos\theta$	$\tan\theta$
$\sqrt{a^2-x^2}$	$x=a\sin\theta, -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$	$dx=a\cos\theta d\theta$	$\frac{x}{a}$	$\frac{\sqrt{a^2-x^2}}{a}$	$\frac{x}{\sqrt{a^2-x^2}}$
$\sqrt{a^2+x^2}$	$x=a\tan\theta, -\frac{\pi}{2}<\theta<\frac{\pi}{2}$	$dx=a\sec^2\theta d\theta$	$\frac{x}{\sqrt{a^2+x^2}}$	$\frac{a}{\sqrt{a^2+x^2}}$	$\frac{x}{a}$
$\sqrt{x^2-a^2}$	$x=a\sec\theta, 0\le\theta<\frac{\pi}{2}$	$dx=a\tan\theta\sec\theta d\theta$	$\frac{\sqrt{x^2-a^2}}{x}$	$\frac{a}{x}$	$\frac{\sqrt{x^2-a^2}}{a}$

Integration of rational functions: Suppose $I=\int\frac{P(x)}{Q(x)}dx$, where $P(x)$, $Q(x)$ are polynomial functions with $Q(x)\not\equiv 0$ and the leading coefficient of $Q(x)$ is 1.
Fact: $Q(x)$ can be factorized into

$$Q(x)=(x-\alpha_1)^{m_1}\cdots(x-\alpha_k)^{m_k}(x^2+\beta_1x+\gamma_1)^{n_1}\cdots(x^2+\beta_1x+\gamma_1)^{n_l},$$

where $x^2+\beta_ix+\gamma_i$ are irreducible polynomials, i.e. they cannot be further factorized as product of linear polynomials. Then perform **partial fraction decomposition**:

$$\frac{P(x)}{Q(x)}=R(x)+\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{A_{i,j}}{(x-\alpha_i)^j}+\sum_{i=1}^{l}\sum_{j=1}^{n_i}\frac{B_{i,j}(x+2\beta_i)+C_{i,j}}{(x^2+\beta_ix+\gamma_i)^j}.$$

$x^2+\beta_ix^2+\gamma_i$'s can written in another form by completing squares:

$$x^2+\beta_ix+\gamma_i=(x-a_i)^2+b_i^2.$$

I can be evaluated by integration term by term: (letting $f_j(x)=((x-a)^2+b^2)^{-j}$)

Term $T(x)$	Method	$\int T(x)dx$
$R(x)$	Direct integration	$\int R(x)dx$
$\frac{1}{(x-\alpha)^j}$	Sub.: $u=x-\alpha$	$\begin{cases}\ln\lvert x-\alpha\rvert+C, & \text{if } j=1\\ \frac{1}{(1-j)(x-\alpha)^{j-1}}+C, & \text{if } j>1\end{cases}$
$\frac{x+2\beta}{(x^2+\beta x+\gamma)^j}$	Sub.: $u=x^2+\beta x+\gamma$	$\begin{cases}\ln\lvert x^2+\beta x+\gamma\rvert+C, & \text{if } j=1\\ \frac{1}{(1-j)(x^2+\beta x+\gamma)^{j-1}}+C, & \text{if } j>1\end{cases}$
$f_j(x)$	$\begin{cases}\text{Sub.: } x=a+b\tan\theta\\ \text{Reduction formula}\end{cases}$	$\begin{cases}\frac{1}{b}\arctan\left(\frac{x-a}{b}\right)+C, & \text{if } j=1\\ \frac{2j-3}{2b^2(j-1)}\int f_{j-1}(x)dx+\frac{x-a}{2b^2(j-1)}f_{j-1}(x), & \text{if } j>1\end{cases}$

t-substitution: Given integrals of the form $I = \int R(\sin x, \cos x, \tan x)dx$, where R is a rational function, we can use substitution $t = \tan\frac{x}{2}$ to change it into the integral

$$\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}\right)\frac{2dt}{1+t^2}$$

of a rational function in t.

In particular, suppose $I = \int \frac{\alpha\sin x+\beta\cos x+\gamma}{a\sin x+b\cos x+c}dx$, where $a, b, c, \alpha, \beta, \gamma$ are real constants.

Case (1): $a = b = 0$. Then $I = \frac{\beta}{c}\sin x - \frac{\alpha}{c}\cos x + \frac{\gamma}{c}x + C$.

Case (2): $a^2 + b^2 > 0 = \alpha = \beta$. By substitution $t = \tan\frac{x}{2}$, $I = \int \frac{2\gamma dt}{(c-b)t^2+2at+b+c}$.

Case (3): $a^2 + b^2 > 0$. We can use t-substitution or find real constants m, n, k such that

$$\begin{aligned} I &= m\int dx + n\int \frac{-b\sin x + a\cos x}{a\sin x + b\cos x + c}dx + k\int \frac{dx}{a\sin x + b\cos x + c} \\ &= mx + n\ln|a\sin x + b\cos x + c| + k\int \frac{dx}{a\sin x + b\cos x + c}, \end{aligned}$$

i.e. solve for (m, n, k) the following system of linear equations:

$$\begin{cases} am - bn &= \alpha, \\ bm + an &= \beta, \\ cm + k &= \gamma. \end{cases}$$

Problems that may be demonstrated in class :

Q1. Suppose a, b are real constants. Let $I_n = \int \frac{dx}{((x-a)^2+b^2)^n}$ for any positive integer n. Prove the following recursive relation: $I_1 = \frac{1}{b}\arctan\left(\frac{x-a}{b}\right) + C$ and

$$I_n = \frac{2n-3}{2b^2(n-1)}I_{n-1} + \frac{x-a}{2b^2(n-1)((x-a)^2+b^2)^{n-1}} \quad \forall n > 1.$$

Q2. Compute the following indefinite/definite integrals:

(a) $\int \frac{x^2+x+1}{\sqrt{x^2-9}}dx$; (b) $\int \sqrt{x^2-4}dx$; (c) $\int \frac{dx}{(x^2+1)^{3/2}}$;

(d) $\int \frac{dx}{x^4+8x^2+16}$; (e) $\int_{-1/2}^{0} \frac{x^6 dx}{(x-2)(x^2+2x-3)}$; (f) $\int \frac{(x^3-7x)dx}{(x-5)^4}$;

(g) $\int_0^{\pi/2} \frac{-\sin x+4\cos x+2}{3\cos x+5}dx$; (h) $\int_{-\pi/4}^{\pi/4} \frac{7\sin x-6\cos x+5\tan x}{3\cos x+2}dx$; (i) $\int \frac{\sin x-1}{\sin x+\tan x}dx$.

Q3. As a variant of t-substitution, for an integral $\int R(\sin^2 x, \cos^2 x)dx$, where R is a rational function, we can make substitution $t = \tan x$. Use this method to evaluate:

(a) $\int_0^{\pi/4} \frac{\sin^2 x-2}{\cos^2 x+1}dx$; (b) $\int_{-\pi/4}^{\pi/4} \frac{dx}{1-\sin^4 x}$; (c) $\int_{-\pi/4}^{\pi/4} \frac{dx}{\tan^2 x+\sec^2 x}$.

Q4. Let a, b be non-zero real constants. Prove that for any positive integer n,

$$\int \frac{dx}{(ae^x+b)^n} = \frac{x}{b^n} - \frac{1}{b^n}\ln|ae^x+b| + \sum_{m=1}^{n-1}\frac{1}{mb^{n-m}(ae^x+b)^m} + C,$$

where empty sum equals zero.