THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010D&E (2016/17 Term 1)
University Mathematics
Tutorial 10 Solutions

Problems that may be demonstrated in class :

Q1. Let $m \in \mathbb{N}$. Evaluate $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^m}{n^{m+1}}$.

Q2. Show that $\ln \frac{n+1}{n} \le \frac{1}{n}$ for any positive integer n. Hence use comparison test to prove that $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Q3. Compute the following indefinite/definite integrals:

(a) $\int x^3 e^{2x} dx$; (b) $\int_1^e x^{-2}(\ln x)^2 dx$; (c) $\int_0^1 x^2 \sin \pi x dx$; (d) $\int x \sin 7x \cos 2x dx$;

(e) $\int \sec^4 x dx$; (f) $\int \sin^2 x \cos^2 x dx$; (g) $\int_0^{\pi/2} \sin^5 x dx$; (h) $\int_{-1}^{1} \sinh x \sec^3 x dx$;

(i) $\int \arcsin x dx$; (j) $\int \sinh x \cos x dx$; (k) $\int e^x \sin 3x dx$; (l) $\int \tan^2 x \sec^3 x dx$.

Q4. $\forall n \in \mathbb{Z}$ with $n \ge 0$, let $I_n = \int (\arcsin x)^n dx$ and $J_n = \int_0^1 (\arcsin x)^n dx$.

(a) Prove that $I_{n+2} = x(\arcsin x)^{n+2} + (n+2)\sqrt{1-x^2}(\arcsin x)^{n+1} - (n+2)(n+1)I_n$;

(b) Prove that

$$J_n = \begin{cases} \sum_{r=0}^{n/2} (-1)^{\frac{n}{2}-r} \frac{n!}{(2r)!} (\frac{\pi}{2})^{2r}, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n+1}{2}} n! + \sum_{r=0}^{(n-1)/2} (-1)^{\frac{n-1}{2}-r} \frac{n!}{(2r+1)!} (\frac{\pi}{2})^{2r+1}, & \text{if } n \text{ is odd.} \end{cases}$$

Solutions :

Q1. Let $f(x) = x^m$ for any $x \in \mathbb{R}$. Then f is continuous on $[0,1]$ and hence integrable over $[0,1]$. By definition of Riemann integrals,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{k^m}{n^{m+1}} = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=1}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}.$$

Q2. Fix a positive integer n. $\frac{1}{x} \le \frac{1}{n}$ for any $x \in [n, n+1]$. Then

$$\ln \frac{n+1}{n} = \int_n^{n+1} \frac{1}{x} dx \le \int_n^{n+1} \frac{1}{n} dx = \frac{1}{n}.$$

Note that

$$\sum_{n=1}^{\infty} \ln \frac{n+1}{n} = \lim_{n\to\infty} \sum_{k=1}^{n} \ln \frac{k+1}{k} = \lim_{n\to\infty} \ln \prod_{k=1}^{n} \frac{k+1}{k} = \lim_{n\to\infty} \ln(n+1) = +\infty.$$

By comparison test, $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to $+\infty$.

Q3. (a)

$$\int x^3 e^{2x}dx = \frac{1}{2}\int x^3 d(e^{2x}) = \frac{1}{2}\left(x^3e^{2x} - 3\int x^2e^{2x}dx\right) = \frac{1}{2}x^3e^{2x} - \frac{3}{4}\int x^2 d(e^{2x})$$
$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}\left(x^2e^{2x} - 2\int xe^{2x}dx\right)$$
$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}\int xd(e^{2x})$$
$$= \frac{1}{2}x^3e^{2x} - \frac{3}{4}x^2e^{2x} + \frac{3}{4}\left(xe^{2x} - \int e^{2x}dx\right)$$
$$= \frac{(4x^3 - 6x^2 + 6x - 3)e^{2x}}{8} + C.$$

(b)

$$\int_1^e x^{-2}(\ln x)^2 dx = -\int_1^e (\ln x)^2 d(x^{-1}) = -\left(\left[x^{-1}(\ln x)^2\right]_1^e - 2\int_1^e x^{-2}\ln x dx\right)$$
$$= -\frac{1}{e} - 2\int_1^e \ln x d(x^{-1}) = -\frac{1}{e} - 2\left(\left[x^{-1}(\ln x)\right]_1^e - \int_1^e \frac{dx}{x^2}\right)$$
$$= -\frac{1}{e} - 2\left(\frac{1}{e} + \left[\frac{1}{x}\right]_1^e\right) = -\frac{1}{e} - 2\left(\frac{1}{e} + \frac{1}{e} - 1\right) = 2 - \frac{5}{e}.$$

(c)

$$\int_0^1 x^2 \sin \pi x dx = -\frac{1}{\pi}\int_0^1 x^2 d(\cos \pi x) = -\frac{1}{\pi}\left(\left[x^2 \cos \pi x\right]_0^1 - 2\int_0^1 x\cos \pi x dx\right)$$
$$= -\frac{1}{\pi}\left(-1 - \frac{2}{\pi}\int_0^1 xd(\sin \pi x)\right)$$
$$= \frac{1}{\pi} + \frac{2}{\pi^2}\left(\left[x\sin \pi x\right]_0^1 - \int_0^1 \sin \pi x dx\right)$$
$$= \frac{1}{\pi} + \frac{2}{\pi^2}\left(0 + \left[\frac{1}{\pi}\cos \pi x\right]_0^1\right) = \frac{1}{\pi} - \frac{4}{\pi^3}.$$

(d)

$$\int x\sin 7x \cos 2x dx = \frac{1}{2}\int x(\sin 9x + \sin 5x)dx$$
$$= -\frac{1}{18}\int xd(\cos 9x) - \frac{1}{10}\int xd(\cos 5x)$$
$$= -\frac{1}{18}\left(x\cos 9x - \int \cos 9x dx\right) - \frac{1}{10}\left(x\cos 5x - \int \cos 5x dx\right)$$
$$= -\frac{x\cos 9x}{18} + \frac{\sin 9x}{162} - \frac{x\cos 5x}{10} + \frac{\sin 5x}{50} + C.$$

(e) Let $u = \tan x$. Then $du = \sec^2 x dx$.

$$\int \sec^4 x dx = \int (\tan^2 x + 1)\cdot \sec^2 x dx = \int (u^2+1)du = \frac{u^3}{3} + u + C$$
$$= \frac{\tan^3 x}{3} + \tan x + C.$$

(f)

$$\int \sin^2 x \cos^2 x dx = \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{2}(1+\cos 2x)dx = \frac{1}{4}\int(1-\cos^2 2x)dx$$
$$= \frac{1}{8}\int(1-\cos 4x)dx = \frac{x}{8} - \frac{\sin 4x}{32} + C.$$

(g)

$$\int_0^{\pi/2} \sin^5 x dx = -\int_0^{\pi/2}(1-\cos^2 x)^2 d(\cos x) = -\int_1^0 (1-u^2)^2 du$$
$$= \int_0^1 (1-2u^2+u^4)du = \left[u - \frac{2}{3}u^3 + \frac{u^5}{5}\right]_0^1 = \frac{8}{15}.$$

(h)

$$\int_{-1}^1 \sinh x \sec^3 x dx = \int_{-1}^0 \sinh x \sec^3 x dx + \int_0^1 \sinh x \sec^3 x dx$$
$$= -\int_{-1}^0 \sinh(-x)\sec^3(-x)dx + \int_0^1 \sinh x \sec^3 x dx$$
$$= -\int_0^{-1} \sinh(-x)\sec^3(-x)d(-x) + \int_0^1 \sinh x \sec^3 x dx$$
$$= -\int_0^1 \sinh y \sec^3 y dy + \int_0^1 \sinh x \sec^3 x dx = 0.$$

(i) Let $u = 1-x^2$. Then $du = -2xdx$.

$$\int \arcsin x dx = x\arcsin x - \int x d(\arcsin x) = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$$
$$= x\arcsin x + \int \frac{1}{2\sqrt{u}}du = x\arcsin x + \sqrt{u} + C$$
$$= x\arcsin x + \sqrt{1-x^2} + C.$$

(j)

$$\int \sinh x \cos x dx = \int \sinh x d(\sin x) = \sinh x \sin x - \int \sin x d(\sinh x)$$
$$= \sinh x \sin x + \int \cosh x d(\cos x)$$
$$= \sinh x \sin x + \cosh x \cos x - \int \sinh x \cos x dx.$$

Rearranging the terms, we get

$$\int \sinh x \cos x dx = \frac{1}{2}(\sinh x \sin x + \cosh x \cos x) + C.$$

(k)

$$\int e^x \sin 3x dx = \int \sin 3x d(e^x) = e^x \sin 3x - 3\int e^x \cos 3x dx$$
$$= e^x \sin 3x - 3\int \cos 3x d(e^x)$$
$$= e^x \sin 3x - 3e^x \cos 3x - 9\int e^x \sin 3x dx.$$

Rearranging the terms, we get

$$\int e^x \sin 3x dx = \frac{e^x}{10}(\sin 3x - 3\cos 3x) + C.$$

(l) Let $I_{m,n} = \int \tan^m x \sec^n x dx$ for any non-negative integers m and n. For $m \geq 2$,

$$\begin{aligned}
I_{m,n} &= \int \tan^{m-2} x(\sec^2 x - 1)\sec^n x dx = I_{m-2,n+2} - I_{m-2,n} \\
&= \frac{1}{m-1}\int \sec^n x d(\tan^{m-1} x) - I_{m-2,n} \\
&= \frac{1}{m-1}\left(\tan^{m-1} x \sec^n x - \int \tan^{m-1} x d(\sec^n x)\right) - I_{m-2,n} \\
&= \frac{\tan^{m-1} x \sec^n x}{m-1} - \frac{n}{m-1} I_{m,n} - I_{m-2,n}.
\end{aligned}$$

Then

$$\begin{aligned}
\frac{m+n-1}{m-1} I_{m,n} &= -I_{m-2,n} + \frac{\tan^{m-1} x \sec^n x}{m-1}, \\
\therefore I_{m,n} &= -\frac{m-1}{m+n-1} I_{m-2,n} + \frac{\tan^{m-1} x \sec^n x}{m+n-1}.
\end{aligned}$$

For $n \geq 2$,

$$\begin{aligned}
I_{m,n} &= \int \tan^m x(\tan^2 x + 1)\sec^{n-2} x dx = I_{m,n-2} + I_{m+2,n-2} \\
&= I_{m,n-2} - \frac{m+1}{m+n-1} I_{m,n-2} + \frac{\tan^{m+1} x \sec^{n-2} x}{m+n-1} \\
&= I_{m,n-2} - \frac{m+1}{m+n-1} I_{m,n-2} + \frac{\tan^{m+1} x \sec^{n-2} x}{m+n-1} \\
&= \frac{n-2}{m+n-1} I_{m,n-2} + \frac{\tan^{m+1} x \sec^{n-2} x}{m+n-1}.
\end{aligned}$$

Now we apply these reduction formulae:

$$\begin{aligned}
I_{2,3} &= -\frac{2-1}{2+3-1} I_{0,3} + \frac{\tan^{2-1} x \sec^3 x}{2+3-1} = -\frac{I_{0,3}}{4} + \frac{\tan x \sec^3 x}{4}, \\
I_{0,3} &= \frac{3-2}{0+3-1} I_{0,3-2} + \frac{\tan^{0+1} x \sec^{3-2} x}{0+3-1} = \frac{I_{0,1}}{2} + \frac{\tan x \sec x}{2}, \\
I_{0,1} &= \int \sec x dx = \ln|\tan x + \sec x| + C, \\
\therefore I_{2,3} &= -\frac{1}{4}\left(\frac{I_{0,1}}{2} + \frac{\tan x \sec x}{2}\right) + \frac{\tan x \sec^3 x}{4} \\
&= -\frac{1}{8}\ln|\tan x + \sec x| - \frac{\tan x \sec x}{8} + \frac{\tan x \sec^3 x}{4} + C. \\
&= \frac{1}{8}\left(\tan x \sec^3 x + \tan^3 x \sec x - \ln|\tan x + \sec x|\right) + C.
\end{aligned}$$

Q4. (a)

$$\begin{aligned}
I_{n+2} =& x(\arcsin x)^{n+2} - \int x d(\arcsin x)^{n+2} \\
=& x(\arcsin x)^{n+2} - (n+2)\int \frac{x}{\sqrt{1-x^2}}(\arcsin x)^{n+1}dx \\
=& x(\arcsin x)^{n+2} + (n+2)\int (\arcsin x)^{n+1} d(\sqrt{1-x^2}) \\
=& x(\arcsin x)^{n+2} + (n+2)\sqrt{1-x^2}(\arcsin x)^{n+1} \\
& - (n+2)\int \sqrt{1-x^2} d((\arcsin x)^{n+1}) \\
=& x(\arcsin x)^{n+2} + (n+2)\sqrt{1-x^2}(\arcsin x)^{n+1} - (n+2)(n+1)I_n.
\end{aligned}$$

(b) Suppose n is any non-negative integer. Let $J_n = \int_0^1 (\arcsin x)^n dx$. By part (a),

$$\begin{aligned}
J_{n+2} &= \left[x(\arcsin x)^{n+2} + (n+2)\sqrt{1-x^2}(\arcsin x)^{n+1}\right]_0^1 - (n+2)(n+1)J_n \\
&= \left(\frac{\pi}{2}\right)^{n+2} - (n+2)(n+1)J_n.
\end{aligned}$$

Note that $J_0 = \int_0^1 dx = 1 = \sum_{r=0}^{0/2}(-1)^{\frac{0}{2}-r}\frac{0!}{(2r)!}(\frac{\pi}{2})^{2r}$ and by Q3(i),

$$\begin{aligned}
J_1 &= \left[x \arcsin x + \sqrt{1-x^2}\right]_0^1 = \frac{\pi}{2} - 1 \\
&= (-1)^{\frac{1+1}{2}} \cdot 1! + \sum_{r=0}^{(1-1)/2}(-1)^{\frac{1-1}{2}-r}\tfrac{1!}{(2r+1)!}(\tfrac{\pi}{2})^{2r+1}.
\end{aligned}$$

Assume for some non-negative integer k,

$$\begin{aligned}
J_{2k} &= \sum_{r=0}^{k}(-1)^{k-r}\frac{(2k)!}{(2r)!}\left(\frac{\pi}{2}\right)^{2r}, \\
J_{2k+1} &= (-1)^{k+1}(2k+1)! + \sum_{r=0}^{k}(-1)^{k-r}\frac{(2k+1)!}{(2r+1)!}\left(\tfrac{\pi}{2}\right)^{2r+1}.
\end{aligned}$$

Then

$$\begin{aligned}
J_{2k+2} &= \left(\frac{\pi}{2}\right)^{2k+2} - (2k+2)(2k+1)J_{2k} \\
&= \left(\frac{\pi}{2}\right)^{2k+2} - (2k+2)(2k+1)\sum_{r=0}^{k}(-1)^{k-r}\frac{(2k)!}{(2r)!}\left(\frac{\pi}{2}\right)^{2r} \\
&= (-1)^{k+1-(k+1)}\frac{(2k+2)!}{(2(k+1))!}\left(\frac{\pi}{2}\right)^{2(k+1)} + \sum_{r=0}^{k}(-1)^{k+1-r}\frac{(2k+2)!}{(2r)!}\left(\frac{\pi}{2}\right)^{2r} \\
&= \sum_{r=0}^{k+1}(-1)^{k+1-r}\frac{(2k+2)!}{(2r)!}\left(\frac{\pi}{2}\right)^{2r}
\end{aligned}$$

and

$$\begin{aligned}
J_{2k+3} &= \left(\frac{\pi}{2}\right)^{2k+3} - (2k+3)(2k+2)J_{2k+1} \\
&= \left(\frac{\pi}{2}\right)^{2k+3} + (-1)^{k+2}(2k+3)! + \sum_{r=0}^{k}(-1)^{k+1-r}\frac{(2k+3)!}{(2r+1)!}\left(\frac{\pi}{2}\right)^{2r+1} \\
&= (-1)^{k+2}(2k+3)! + \sum_{r=0}^{k+1}(-1)^{k+1-r}\frac{(2k+3)!}{(2r+1)!}\left(\frac{\pi}{2}\right)^{2r+1}.
\end{aligned}$$

By mathematical induction, we have for any non-negative integer k,

$$\begin{aligned}
J_{2k} &= \sum_{r=0}^{k}(-1)^{k-r}\frac{(2k)!}{(2r)!}\left(\frac{\pi}{2}\right)^{2r}, \\
J_{2k+1} &= (-1)^{k+1}(2k+1)! + \sum_{r=0}^{k}(-1)^{k-r}\frac{(2k+1)!}{(2r+1)!}\left(\tfrac{\pi}{2}\right)^{2r+1},
\end{aligned}$$

which implies that for any non-negative integer n,

$$J_n = \begin{cases} \sum_{r=0}^{n/2}(-1)^{\frac{n}{2}-r}\frac{n!}{(2r)!}(\frac{\pi}{2})^{2r}, & \text{if } n \text{ is even;} \\ (-1)^{\frac{n+1}{2}}n! + \sum_{r=0}^{(n-1)/2}(-1)^{\frac{n-1}{2}-r}\frac{n!}{(2r+1)!}(\frac{\pi}{2})^{2r+1}, & \text{if } n \text{ is odd.} \end{cases}$$