Power Series.

Intro. The main object of study in Calculus are functions.

Basically, a function f (or $f(x)$ if one wants to emphasize the variable) is a rule

$$f: x \longmapsto f(x) \quad \text{or} \quad x \overset{f}{\longmapsto} f(x)$$

It requires: Each input x has a unique output $f(x)$.

also known as "value"

Examples ① Functions such as polynomials, $\sin(x)$, $\cos(x)$, e^x, $\ln x$. Functions built up from them via $+, -, \times, \div, \circ, \sqrt[n]{\ }$ etc. in finite steps

② Functions defined by integrals

e.g. $$G(x) = \int_a^x e^{\alpha t}\,dt$$

③ Power series (or "functions defined by power series")

Def. (Power Series)

A power series centered at c is an expression of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k$$

Rmk: ① The rule $$x \overset{f}{\longmapsto} \sum_{k=0}^{\infty} a_k (x-c)^k$$

is a function.

② $f(c) = 0$

③ Question: What is the maximal domain of $f(x)$?

Answer: Basically, it is the set

$$(c-R,\ c+R)$$

where $R = \frac{1}{\rho}$ & $\rho = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$.

Convention: $\frac{1}{\infty} = 0$, $\frac{1}{0} = \infty$

Examples ① $f(x) = 1 + x + x^2 + \cdots + x^k + \cdots = \sum_{k=0}^{\infty} x^k$

$\uparrow$ general term

$$\rho = \lim_{k\to\infty} \frac{1}{1} = 1 \Rightarrow R = 1$$

Max domain $= (0-1, 0+1) = (-1, 1)$.

② $f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. $\rho = \lim_{k\to\infty} \frac{1/(k+1)!}{1/k!} = \lim_{k\to\infty} \frac{1}{k+1} = 0$

$\therefore$ $R = \infty$ & max. domain $= (-\infty, +\infty)$ or $\mathbb{R}$.

③ $f(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} (x-1)^k$ is centered at 1 & has $R = 1$

Rmk:

① Q: Is the power series finite (or "convergent") or infinite / not defined ("divergent") at the 2 end-points?

A: This is a difficult question and has to be checked case by case.

Properties of Power Series

① Given 2 power series $\sum_{k=0}^{\infty} a_k (x-c)^k$, $\sum_{k=0}^{\infty} b_k (x-c)^k$, one can perform the operations $+, -, \times, \div$ with them.

E.g. • $\frac{x}{1-x} = x(1 + x + x^2 + \cdots + x^k + \cdots)$

$= x + x^2 + x^3 + \cdots + x^{k+1} + \cdots = \sum_{k=1}^{\infty} x^k$

• $\frac{1+x+x^2}{1-x} = (1+x+x^2)(1+x+x^2+\cdots+x^k+\cdots)$

$= 1 + (1+1)x + (1+1+1)x^2 + \cdots$

$\uparrow$ constant term $\quad \uparrow$ x^1 term $\quad \uparrow$ x^2 term

② Differentiation & Integration of power series

(i) Let $\sum_{k=0}^{\infty} a_k (x-c)^k$ be a power series on $(c-R, c+R)$

then its derivative, i.e. $\sum_{k=1}^{\infty} a_k \cdot \frac{d(x-c)^k}{dx}$ is also a power

series on $(c-R, c+R)$.

(ii) Similar result holds for indefinite integral.

Applications:

① Q: Find the power series of $\tan^{-1} x$ centered at 0.

A:

Since $y = \tan^{-1} x$ gives $\tan y = x \Rightarrow \sec^2 y \cdot y' = 1$

$$\Rightarrow y' = \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

$\therefore$ the power series of $\frac{d \tan^{-1} x}{dx}$ is $\frac{1}{1-(-x^2)}$

$$= 1 + (-x^2) + x^4 - x^6 + x^8 + \cdots + (-1)^k x^{2k} + \cdots$$

Integrating this gives

$$\int \frac{d \tan^{-1} x}{dx} dx = \tan^{-1} x + C = \sum_{k=0}^{\infty} (-1)^k \int x^{2k} dx$$

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

To find C, we let $x=0$ (or any other convenient value) to get

$$\tan^{-1} 0 + C = \sum_{k=0}^{\infty} (-1)^k \frac{0^{2k+1}}{2k+1}$$

$$\Rightarrow C = 0$$

Concl: $\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$