MATH 1510 Notes
Week 1

1. Introduction

In this course, we plan to discuss what is known as 'infinitesimal' calculus. The main object of study in this course are functions.

Roughly speaking, a function is described by the following

Definition 1.1. A function is a rule, f, which assigns a unique value, $f(x)$, to each given input value x.

Remark 1. In the above definition, we have 3 things: (i) a rule, denoted by a single letter f, (ii) a value $f(x)$ (for each x) and (iii) an input x (also known as 'variable' or more precisely 'independent variable').

- The rule can be denoted by a single or multiple letter(s) (but without the brackets and the input variable!). Examples are: $f, g, h, \cos, \sin, \tan$
- The value $f(x)$ is also known as 'evaluation' of f at x.
- Often times, we denote a function by the following schematic diagram:
  $$x \overset{f}{\mapsto} f(x)$$
  Sometimes, we also use the following schematic diagram:
  $$f:\ x \mapsto f(x).$$

After defining a function, we go on to mention its domain.

In this course, when we talk about domain, we usually mean 'natural domain', i.e. the 'largest possible domain'.

Example 1. Let f be the function given by the rule $f(x) = 1/x$ (some people would write it more rigorously as:

$$f(x) := 1/x,$$

where the expression on the left-hand side of the $:=$ sign is 'evaluated' or 'computed' by the formula on the right-hand side. Then the domain (i.e. the nat. dom. (or natural domain) is the set

$$\{x \in \mathbb{R} \mid x \neq 0\}.$$

Remark 2. In the above example, we have used notation for a set. Note that it consists of the following:

(1) Two curly brackets;
(2) the symbol of element in the set, i.e. 'x' followed by '$\in \mathbb{R}$' (this may be omitted if we know that it is a real no.;
(3) a vertical bar $|$, to the right of which we write down further properties the element x must satisfy (in order to be in this set).

1.1. **Sequences.** A sequence is a function, whose domain is the set of positive integers, denoted by $\mathbb{N}^*$.

Remark 3. In the lecture, we mentioned that there are two ways of writing down a set, either by specifying the 'properties' an element in the set has to satisfy, or by listing all the elements in the set. E.g.
$\{x|\ x \text{ is a positive integer}\}$ versus $\{1, 2, 3, \cdots\}$.

The first one is convenient for defining a set consisting of infinitely many elements, because in such cases, it may not always be possible to list all of them.

Example 2. Let f be the rule given by $f(x) = 1/x$ and now take the domain to be $\mathbb{N}^*$, then we get a sequence

$$n \overset{f}{\mapsto} 1/n$$

given by $f(n) = 1/n$. Here we changed the notation for the input from x to n to remind ourselves that the input is a positive integer.

Traditionally, people denoted sequences by
(1) $\{f_n\}$ (which is a short form of $\{f_n|\ n \in \mathbb{N}^*\}$ or $\{f_1, f_2, f_3, \cdots\}$. (when we use curly brackets, we are thinking of a set), or
(2) (f_n) (which is a short form of $(f_1, f_2, f_3, \cdots)$ (when we use round brackets, we are thinking of coordinates.)

2. Examples of Functions – Trigonometric Functions

Before we can work with functions, we have to have examples of them. From school math, we know the following functions, the first three of them were defined already by ancient Greeks.

$$\sin, \cos, \tan, \exp, \ln$$

Their main properties are
(1) periodic, i.e. their graphs are cut-and-pasting of some fundamental pieces, in terms of formulas, this is given by

$$\sin(x + 2\pi) = \sin(x) \quad \text{for each} \quad x$$

$$\cos(x + 2\pi) = \cos(x) \quad \text{for each} \quad x$$

$\tan(x)$ is slightly different.

(2) sine and cosine functions are bounded above by $+1$ and below by -1. This is the consequence of the Pythagoras Theorem:

$$\cos^2 x + \sin^2 x = 1 \quad \text{for each} \quad x$$

2.1. **Some useful formulas.** Here are some useful formulas for sine, cosine and tangent (which holds for each x in their natural domains).

(1) $\sin^2 x + \cos^2 x = 1$, $1 + \tan^2 x = \sec^2 x$
(2) $\sin(x \pm y) = \sin(x)\cos(x) \pm \sin(y)\cos(x)$
(3) $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
(4) double angle formulas for sine and for cosine
(5) formula for $\sin(A) + \sin(B)$ in terms of products of sine and cosine and other similar formulas (these are derivable form (2) and (3).

Remark 4. If one assumes that the formula

$$e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

continues to hold if y is replaced by the imaginary number $\sqrt{-1}x$, where x is a real number, then one can obtain the formulas (2) and (3) above easily (where one has also to assume the following formulas for sine and cosine which were discovered by Euler):

(1) $\sin x = x - x^3/3! + x^5/5! - \cdots$
(2) $\cos x = 1 - x^2/2! + x^4/4! - \cdots$

These formulas were also discovered by Indian astronomers around the 15th century. It is important to remember that these formulas assume that x is measured in 'radians'.

2.2. **A remark on our goal.** One main idea in calculus is 'approximation'. Indeed, the above formulas for sine and cosine can be explained by the Taylor's Theorem (which we will explain in the middle of the course). This theorem roughly says:

A function (with $f', f'', f''', \cdots$ all defined) can be written as

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \cdots$$

One can make use of this formula to calculate 'approximate' value of $\sin(x)$ or $\cos(x)$ for any given x by considering suitable number of terms (instead of 'all' terms, which a computer cannot handle!)

3. Limit of sequences

Before we can discuss Taylor's Theorem (which is still a long way ahead), one has to first discuss 'limits', because 'limits' is a very important tool in understanding

(1) functions,
(2) sequences,
(3) derivatives of functions

We first begin with $+, -, \times, \div$ of limits of sequences. We have

Theorem 3.1. *Let $\{a_n\}$ and $\{b_n\}$ be two sequences with limits L and M respectively (suppose also that L and M are finite numbers). Then we have*

(1) $\lim_{n\to\infty}(a_n + b_n) = L + M$,
(2) $\lim_{n\to\infty}(a_n - b_n) = L - M$,
(3) $\lim_{n\to\infty}(a_n \times b_n) = LM$,
(4) $\lim_{n\to\infty}(a_n / b_n) = L/M$,

In the last item, one has to assume also that $M \neq 0$.

Using this and some well-known simple facts such as $\lim_{n\to\infty} 1/n^k = 0, k \in \mathbb{N}^*$ and $\lim_{n\to\infty} n^k = \infty$, together with the following 'Monotone Convergence Theorem and 'Sandwich Theorem', one can compute a lot of limits of sequences.

Theorem 3.2. *Let $\{a_n\}$ be a sequence satisfying*

(1) $a_n \leq a_{n+1}$ *for each* $n = 1, 2, 3, \cdots$,
(2) $a_n \leq C$ *for some real number* C

then $\lim_{n\to\infty} a_n$ *exists (and is a finite number).*

Theorem 3.3. *Let $\{a_n\}$, $\{b_n\}$ $\{c_n\}$ be three sequences satisfying*

(1) $a_n \leq b_n \leq c_n$ *for each positive integer* n,
(2) $\lim_{n\to\infty} a_n = L$ *and* $\lim_{n\to\infty} c_n = L$

then $\lim_{n\to\infty} b_n$ *exists (and is equal to L).*