Week 7
Taylor Series

Definition.

Given a function f which is infinitely differentiable at a (i.e. $f^{(k)}(a)$ is defined for $k = 0, 1, 2, 3, \ldots$). The **Taylor series of f (centered) at a** is the power series:

$$\begin{aligned} T(x) &= \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots \end{aligned}$$

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In general, for any power series of the form $S(x) = \sum_{k=0}^{\infty} a_k(x-a)^k$, the value of S at any given $c \in \mathbb{R}$ is by definition the limit:

$$S(c) := \lim_{n\to\infty} \sum_{k=0}^{n} a_k(c-a)^k.$$

Note that this limit does not necessarily exist. If it does exist, we say that the power series S **converges** at $x = c$, otherwise we say that it **diverges** at $x = c$.

Example.

The Taylor series at $a = 0$ for various functions f are as follows:

$f(x)$	$P(x)$
$\cos x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$
$\sin x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$
e^x	$\sum_{k=0}^{\infty} \frac{x^k}{k!}$
$\ln(1+x)$	$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$
$\arctan x$	$\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$
$\frac{1}{1-x}$	$\sum_{k=0}^{\infty} x^k$

Example.

The Taylor $T(x)$ series of $f(x) = e^x$ at $a = 0$ converges everywhere. Moreover, for each $x \in \mathbb{R}$, we do have:

$$T(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x.$$

Similarly, for all $x \in \mathbb{R}$, we have:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = \sin x$$

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = \cos x$$

However,

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The Taylor series of $f(x) = \ln(1+x)$ at $a = 0$ is:

$$T(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} x^k,$$

which converges only for $x \in (-1, 1]$.
For such x we do have:

$$T(x) = f(x).$$

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In particular, we have:

$$\ln 2 = \ln(1+1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} 1^k = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Remark.

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There are functions whose Taylor series converge everywhere, but not to the functions themselves.

Shortcuts for Computing Taylor Series

Theorem.

Let $S(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$ be a power series which converges on an open interval of the form $(a-r, a+r)$, $r > 0$, then the function $S(x)$ is differentiable on $(a-r, a+r)$, with $S'(x) = \sum_{k=0}^{\infty} k a_k (x-a)^{k-1}$ for all $x \in (a-r, a+r)$.

Applying this theorem repeatedly, it may be shown that $S(x)$ is in fact infinitely differentiable on $(a-r, a+r)$, and its Taylor series at $x = a$ is itself. That is:

$$\frac{S^{(k)}(a)}{k!} = a_k, \quad k = 0, 1, 2, \ldots.$$

Put differently:

Corollary.

Let f be a function. If there is a sequence $\{a_k\}_{k=0}^{\infty}$ such that:

$$f(x) = \sum_{k=0}^{\infty} a_k (x-a)^k$$

for all x in some open interval centered at a, then $\sum_{k=0}^{\infty} a_k (x-a)^k$ is the Taylor series of f at $x = a$, with $a_k = \dfrac{f^{(k)}(a)}{k!}$.

Corollary.

If:

$$\sum_{k=0}^{\infty} a_k (x-a)^k = \sum_{k=0}^{\infty} b_k (x-a)^k$$

for all x in some open interval centered at a, then $a_k = b_k$ for all k.

Exercise.

Find the Taylor series of f at the given point a.

$f(x)$	a
$\sin(5x)$	0
$x^3 \cos x$	0
$\sin(x-\pi)$	π
$\ln x$	1
$\dfrac{1}{2-x}$	0
$\dfrac{1}{1+x}$	0
$\dfrac{1}{1+x^2}$	0
$\arctan x$	0
$\dfrac{x+1}{x^2+x+1}$	0