Week 5
Mean Value Theorem
L'Hôpital's Rule

Higher Order Derivatives

Let f be a function.

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Its derivative f' is often called the **first derivative** of f.

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The derivative of f', denoted by f'', is called the **second derivative** of f.

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If $f''(c)$ exists, we say that f is **twice differentiable** at c.

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For $n \in \mathbb{N}$, the n-th derivative of f, denoted by $f^{(n)}$ is defined as the derivative of the $(n-1)$-st derivative of f.

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If $f^{(n)}(c)$ exists, we say that f is n times differentiable at c.

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We sometimes consider f to be the "zero"-th derivative of itself, i.e. $f^{(0)} := f$.

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In the Leibniz notation, we have:

$$f^{(n)}(x) = \underbrace{\frac{d}{dx}\frac{d}{dx}\cdots\frac{d}{dx}}_{n \text{ times}} f,$$

which is customarily written as:

$$\frac{d^n f}{dx^n}\ .$$

Exercise.

Let:

$$f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Find $f''(0)$, if it exists.

Theorem.

General Leibniz Rule. Let $n \in \mathbb{N}$. Given any functions f, g which are n times differentiable at c, their product fg is also n times differentiable at c, with:

$$(fg)^{(n)}(c) = \sum_{k=0}^{n} C_k^n f^{(k)}(c) g^{(n-k)}(c)$$

Notice that when $n = 1$ this rule is simply the product rule we have introduced before.

The Mean Value Theorem

Theorem.

Rolle's Theorem Let $f : [a, b] \longrightarrow \mathbb{R}$ be a function which is continuous on $[a, b]$ and differentiable on (a, b) (i.e. $f'(x)$ exists for all $x \in (a, b)$). If $f(a) = f(b)$, then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof.

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Sketch of Proof. First, it follows from the Extreme Value Theorem that f has an absolute maximum or minimum at a point c in (a, b). It may then be shown that:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0,$$

using that fact that if $f(c)$ is an absolute extremum, then $\dfrac{f(c+h) - f(c)}{h}$ is both ≤ 0 and ≥ 0.

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Theorem.

The Mean Value Theorem (MVT). If a function $f : [a, b] \longrightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Proof.

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Let f be a function which satisfies the conditions of the theorem. Define a function $g : [a, b] \longrightarrow \mathbb{R}$ as follows:

$$g(x) = f(x) - \left[\left(\frac{f(b) - f(a)}{b - a}\right)(x - a) + f(a)\right], \quad x \in [a, b].$$

(Intuitively, g is obtained from f by subtracting from f the line segment joining $(a, f(a))$ and $(b, f(b))$.) Observe that:

$$g(a) = g(b) = 0,$$

so the function g satisfies the conditions of Rolle's Theorem. Hence, there exists $c \in (a, b)$ such that:

$$0 = g'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

which implies that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

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Exercise.

Using the mean value theorem to prove that for $x > 0$,

$$\frac{x}{1 + x} < \ln(1 + x) < x.$$

Hence, deduce that for $x > 0$,

$$\frac{1}{1 + x} < \ln\left(1 + \frac{1}{x}\right) < \frac{1}{x}.$$

Applications of the Mean Value Theorem

Theorem.

Let f be a differentiable function on an open interval (a,b). If $f'(x)=0$ for all $x \in (a,b)$, then f is constant (a,b).

Proof.

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Exercise. For any $x_1, x_2 \in (a,b)$, show that the difference $f(x_2)-f(x_1)$ is equal to 0.

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Theorem.

Let f be a differentiable function on an open interval (a,b). If $f'(x)>0$ (resp. $f'(x)<0$) for all $x \in (a,b)$, then f is increasing (resp. decreasing) on (a,b).

Proof.

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We will prove the case $f'(x)>0$.
Suppose $f'(x)>0$ for all $x \in (a,b)$. Given any $x_1, x_2 \in (a,b)$, such that $x_1 < x_2$, by the MVT there exists $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2)-f(x_1)}{x_2 - x_1}.$$

By the condition $f'(x)>0$ for all $x \in (a,b)$, we have $f'(c)>0$. Also, $x_2 - x_1 > 0$. Hence, $f(x_2)-f(x_1)>0$. This shows that f is increasing on (a,b).

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Corollary.

First Derivative Test Let $f : A \longrightarrow \mathbb{R}$ be a continuous function. For $c \in A$, if there exists an open interval (a,b) containing c such that $f'(x)<0$ (in particular it exists) for all $x \in (a,c)$, and $f'(x)>0$ for all $x \in (c,b)$, then f has a local minimum at c. Similarly, if $f'(x)>0$ for all $x \in (a,c)$ and $f'(x)<0$ for all $x \in (c,b)$, then f has a local maximum at c.

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Note: In the special case that the domain of f is an open interval (a, b), if $f'(x) > 0$ for all $x \in (a, c)$, and $f'(x) < 0$ for all $x \in (c, b)$, then f has an absolute maximum at c.
Similarly f has an absolute minimum at c if each of the above inequalities is reversed.

Exercise.

$f(x) = x^{\frac{1}{3}} - \frac{1}{3}x - \frac{2}{3}$ for $x > 0$. Show that $f(x) \leq 0$ for all $x > 0$. Then, deduce that:

$$u^{\frac{1}{3}}v^{\frac{2}{3}} \leq \frac{1}{3}u + \frac{2}{3}v$$

for u, $v > 0$.

Theorem.

Second Derivative Test Let f be a function twice differentiable at $c \in \mathbb{R}$, such that $f'(c) = 0$. If:

- $f''(c) > 0$, then f has a local minimum at c.
- $f''(c) < 0$, then f has a local maximum at c.

Proof.

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Sketch of Proof. Suppose $f''(c) > 0$, by the definition of $f''(c)$ as the derivative of f' at c, we have:

$$0 < f''(c) = \lim_{h \to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h \to 0} \frac{f'(c+h)}{h}.$$

It follows from the above identity that $f'(c+h)$ is > 0 for sufficiently small positive h, and < 0 for sufficiently small negative h.

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Hence there is an open interval (a, b) containing c such that f' is negative on (a, c) and positive on (c, b). So, f has a local minimum at c by the First Derivative Test.

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The case $f''(c) < 0$ may be proved similarly.

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Theorem.

Cauchy's Mean Value Theorem. If $f, g : [a, b] \longrightarrow \mathbb{R}$ are functions which are continuous on $[a, b]$ and differentiable on (a, b), and $g(a) \neq g(b)$, then there exists $c \in (a, b)$ such that:

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

Proof.

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Exercise. Apply Rolle's Theorem to:

$$h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$$

Theorem.

L'Hôpital's Rule. Let $c \in \mathbb{R}$. Let $I = (a, b)$ be an open interval containing c. Let f, g be functions which are differentiable at every point in $(a, c) \cup (c, b)$. Suppose:

- $\lim_{x \to c} f(x)$ and $\lim_{x \to c} g(x)$ are both equal to 0 or both equal to $\pm\infty$.
- $g'(x) \neq 0$ for all $x \in (a, c) \cup (c, b)$.
- $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists.

Then,

$$\boxed{\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.}$$

Exercise.

Use l'Hôpital's rule to evaluate the following limits:

1. $\lim_{x \to 0} \dfrac{1 - x \cot x}{x \sin x}$

2. $\lim_{x \to 0^+} x^{\frac{1}{1+\ln x}}$

3. $\lim_{x \to +\infty} x \left(\dfrac{\pi}{2} - \tan^{-1} x \right)$

4. $\lim_{x \to +\infty} (e^x + x)^{\frac{1}{x}}$