Week 4
Differentiation

Derivatives

Definition.

We say that a function f is **differentiable** at c if the limit:

$$f'(c) := \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

exists. The limit $f'(c)$, if it exists, is called the **derivative** of f at c.

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We say that a function f is **differentiable** if it is differentiable at every point in its domain.

Exercise.

Let $f(x) = |x|$. Is f differentiable at $x = 0$? If so, find $f'(0)$.

Theorem.

If a function f is differentiable at c, then it is also continuous at c.
(The converse is false in general.)

Exercise.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^3 & \text{if} \quad x \leq 1; \\ ax + b & \text{if} \quad x > 1. \end{cases}$$

Suppose $f(x)$ differentiable at $x = 1$, find the values of a and b.

Tangent Line

If the derivative $f'(c)$, if it exists, then there exists a tangent line to the graph $y = f(x)$ of f at $(c, f(c))$.

Moreover, the slope of the tangent line is $f'(c)$, and the tangent line is the graph of the equation:

$$y = f'(c)(x - c) + f(c).$$

Given $f : A \longrightarrow \mathbb{R}$, the correspondence $x \mapsto f'(x)$ defines the **derivative function** $f' : A' \longrightarrow \mathbb{R}$, where A' is the set of all points $c \in A$ at which f is differentiable.

Some Common Derivative Identities:

$f(x)$	$f'(x)$
constant	0
$ax + b \quad (a, b \in \mathbb{R})$	a
$x^n \quad (n \in \mathbb{Z},\ n \neq 0, 1)$	nx^{n-1}
$x^r \quad (r \in \mathbb{R},\ n \neq 0, 1;\ x > 0)$	rx^{r-1}
e^x	e^x
$\ln x$	$\frac{1}{x}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\sec x$	$\sec x \tan x$
$\cot x$	$-\csc^2 x$
$\csc x$	$-\csc x \cot x$
$\arctan x$	$\frac{1}{x^2 + 1}$
$\arcsin x \quad (-1 < x < 1)$	$\frac{1}{\sqrt{1 - x^2}}$

Leibniz Notation

If f is defined in terms of an independent variable x, we often denote $f'(x)$ by $\frac{df}{dx}$. Under this notation, for a given $c \in \mathbb{R}$ the value $f'(c)$ is denoted by:

$$\left.\frac{df}{dx}\right|_{x=c}$$

Rules of Differentiation

Let f, g be functions differentiable at $c \in \mathbb{R}$. Then:

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Sum/Difference Rule

$f \pm g$ is differentiable at c, with:

$$(f \pm g)'(c) = f'(c) \pm g'(c).$$

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Product Rule

fg is differentiable at c, with:

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

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Quotient Rule

f/g is differentiable at c provided that $g(c) \neq 0$, in which case we have:

$$\left(\frac{f}{g}\right)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}.$$

Chain Rule

Theorem.

Suppose f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c, with:

$$(g \circ f)'(c) = g'(f(c))f'(c).$$

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In the Leibniz notation, the chain rule says that if f is a differentiable function of u and u is a differentiable function of x, then:

$$\frac{df}{dx} = \frac{df}{du}\frac{du}{dx},$$

$$\left.\frac{df}{dx}\right|_{x=c} = \left.\frac{df}{du}\right|_{u=u(c)} \left.\frac{du}{dx}\right|_{x=c}$$

Exercise.

Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

Find f'. Is f' continuous at $x = 0$?

Implicit Differentiation

Example.

For $x > 0$,

$$\frac{d}{dx} \ln x = \frac{1}{x}.$$

Proof.

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Consider the equation:

$$e^{\ln x} = x$$

Differentiating both sides with respect to x, and applying the Chain Rule, we have:

$$\begin{aligned} \frac{d}{dx} e^{\ln x} &= \frac{d}{dx} x \\ \underbrace{e^{\ln x}}_{=x} \frac{d}{dx} \ln x &= 1 \end{aligned}$$

Hence, $\dfrac{d}{dx} \ln x = \dfrac{1}{x}$.

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Exercise.

Consider the curve $C : y^4 - y\cos(x) - x^4 = 0$.

1. Find $\dfrac{dy}{dx}$. Express your answer in terms of x, y only.
2. Let $P = \left(\frac{\pi}{2}, -\frac{\pi}{2}\right)$.
   - Verify that the point P lies on the curve C.
   - Find the equation of the normal to the curve C at the point P.

Theorem.

Let f be an injective function differentiable at $x = c$. If $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$, with:

$$\left(f^{-1}\right)'(f(c)) = \frac{1}{f'(c)}.$$

Example.

Consider the injective function:

$$f : [-\pi/2, \pi/2] \longrightarrow \mathbb{R},$$

$$f(x) = \sin x, \quad x \in [-\pi/2, \pi/2].$$

Then, $f'(x) = \cos x$ for $x \in (-\pi/2, \pi/2)$. In particular, f' is nonzero on $(-\pi/2, \pi/2)$. The inverse of f is:

$$f^{-1} = \arcsin : [-1, 1] \longrightarrow [-\pi/2, \pi/2].$$

For any $y \in (-1, 1)$, we have $y = f(x) = \sin(x)$ for a unique $x = \arcsin y$ in $(-\pi/2, \pi/2)$. Hence,

$$\arcsin' y = \left(f^{-1}\right)'(y) = \left(f^{-1}\right)'(f(x)) = \frac{1}{f'(x)} = \frac{1}{\cos x}.$$

By the Pythagorean Theorem, we know that:

$$\cos x = \pm\sqrt{1 - \sin^2 x}\ .$$

Moreover, since $x \in (-\pi/2, \pi/2)$, we have $\cos x > 0$, so:

$$\cos x = +\sqrt{1 - \sin^2 x} = \sqrt{1 - \sin^2(\arcsin(y))} = \sqrt{1 - y^2}.$$

In conclusion, for $y \in (-1, 1)$, we have:

$$\arcsin' y = \frac{1}{\sqrt{1 - y^2}}.$$