Week 3
Limits
Continuity

Sandwich Theorem for Functions on the Real Line

Theorem.

Let $a \in \mathbb{R}$, A an open neighborhood of a which does not necessarily contain a itself. Let $f, g, h : A \longrightarrow \mathbb{R}$ be functions such that:

$$g(x) \leq f(x) \leq h(x) \qquad \text{for all } x \in A,$$

and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L.$$

Then, $\lim_{x \to a} f(x) = L$.

Similary,

Theorem.

If f, g, h are functions on $\mathbb{R}$ such that:

$$g(x) \leq f(x) \leq h(x)$$

for all x sufficiently large, and

$$\lim_{x \to \infty} g(x) = \lim_{x \to \infty} h(x) = L,$$

then $\lim_{x \to \infty} f(x) = L$.

Exercise.

Find the following limits, if they exist:

- $\lim_{x \to \infty} \dfrac{\sin x}{x}$

- $\lim_{x \to \infty} \dfrac{x + \sin x}{x - \sin x}$

Theorem.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

+

Corollary.

$$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}.$$

Exercise.

Find the following limits, if they exist:

- $\lim_{x \to 0} \frac{\sin(5x)}{\tan(3x)}$
- $\lim_{x \to 0} \frac{x^3 \cos\left(\frac{1}{x}\right)}{\tan x}$

Theorem.

$$\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = \lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e$$

+

Corollary.

$$\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = \lim_{x \to 0} (1 - x)^{\frac{1}{x}} = \frac{1}{e}$$

Exercise.

Find:

$$\lim_{x \to \infty} \left(\frac{x+1}{x-1}\right)^x$$

Definition.

For each $x \in \mathbb{R}$, we let:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Note that:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n.$$

Theorem.

For all $n \in \{1, 2, 3, \ldots\}$, we have:

$$\lim_{x \to \infty} \frac{x^n}{e^x} = 0.$$

+

Corollary.

For all $n \in \{1, 2, 3, \ldots\}$, and $b > 1$, we have:

$$\lim_{x \to \infty} \frac{x^n}{b^x} = 0.$$

Definition.

A function $f : A \longrightarrow \mathbb{R}$ is said to be **continuous** at $c \in A$ if:

$$\lim_{x \to c} f(x) = f(c).$$

A function is said to be **continuous** if it is continuous at every point in its domain.

Should c be an endpoint in the domain of f, the continuity of f at c is defined in terms of a one-sided limit.
That is, right limit if c is a left endpoint, and left limit if c is a right endpoint. +

Hence, the function:

$$f(x) = \sqrt{x}$$

is continuous at $x = 0$, since $\mathrm{Domain}(f) = [0, \infty)$, and:

$$\lim_{x\to 0^+} f(x) = 0 = f(0).$$

The following "elementary functions" are continuous at every element in their domains:

$$f(x) = x, \frac{1}{x}, \sin x, \cos x, \tan, e^x, \ln x, \arcsin x, \arccos x, \arctan x$$

Due to the laws of sum/difference/product/quotient for limits, the sum/difference/product/quotient of continuous functions is also continous.

In particular, polynomials and rational functions are all continuous on their domains.

Theorem.

for functions g, f with the property that $\lim\limits_{x\to a} g(x)$ exists and f is continuous at $\lim_{x\to a} g(x)$, we have:

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right).$$

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Example.

It follows from this theorem that:

$$\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$$

And from this may be further deduced that:

$$\lim_{x\to 0} \frac{e^t - 1}{t} = 1.$$

It also follows from the previous theorem that any composite of continuous functions is continuous.

Example.

The following functions are all continuous, since they are the sums, differences, products, quotients, or composites of other continuous functions:

$$f(x) = \frac{e^{\cos\left(\frac{1}{x}\right)}}{x^7 - 9x^2 + 23}$$
$$g(x) = \frac{1}{\arctan x} - \sqrt[3]{\log_5(2^x + 1)}$$
$$h(x) = \sin\left(x^{-3} + \left(\cos\left(e^{x^2} + 1\right)\right)\right)$$

Example.

The following functions are continuous at every point on the real line:

- $$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0; \end{cases}$$
- $$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{e^x - 1}\right), & x \neq 0; \\ 0, & x = 0; \end{cases}$$

Exercise.

Let $f : \mathbb{R} \to \mathbb{R}$ be a function that satisfies

- $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$;
- $f(x)$ is continuous at $x = 0$ and $f(0) \neq 0$.

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1. Show that $f(0) = 1$.
2. Show that $f(x)$ is continuous on $\mathbb{R}$.

Theorem.

Intermediate value Theorem (IVT). If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then f attains every value between $f(a)$ and $f(b)$.

In other words, for any $y \in \mathbb{R}$ between the values of $f(a)$ and $f(b)$, there exists $c \in [a, b]$ such that $f(c) = y$.

Exercise.

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- Show that $f(x) = x^5 + x^2 - 10 = 0$ has a real root between $x = 1$ and $x = 2$.
- Show that the range of $f(x) = e^x - \sqrt{x}$ *contains* $[1, \infty)$.

Consider a function $f : A \longrightarrow \mathbb{R}$.

Definition.

- If there is an element $c \in A$ such that: $f(c) \leq f(x)$ for all $x \in A$, we say that $f(c)$ is the (absolute) **minimum** of f.
- If there is an element $d \in A$ such that: $f(d) \geq f(x)$ for all $x \in A$, we say that $f(d)$ is the (absolute) **maximum** of f.

Note that in general a maximum or minimum does not necessarily exist. However: +

Theorem.

Extreme Value Theorem (Maximum-Minimum Theorem). If f is a continuous function defined on a closed interval $[a, b]$, then it does attain both a maximum and a minimum on $[a, b]$.

In general (for any real-valued function f),

Definition.

- If $f(c) \geq f(x)$ for all x in an open interval containing c, we say that f has a **local/relative maximum** at c.
- If $f(c) \leq f(x)$ for all x in an open interval containing c, we say that f has a **local/relative minimum** at c.
