Week 10 - 11
Indefinite Integrals
Partial Fractions

Miscellaneous Examples

Evaluate:

- $\displaystyle\int \tan^4 x \sec x \, dx.$
- $\displaystyle\int \arccos x \, dx.$
- $\displaystyle\int f(x)\, dx$, where:

$$f(x) = \begin{cases} x^3 + 2, & x > 1 \\ 2x + 1, & x \leq 1. \end{cases}$$

Definition.

A rational function $\frac{r}{s}$, where r, s are polynomials, is said to be **proper** if:

$$\deg r < \deg s.$$

By performing long division of polynomials, any rational function $\frac{p}{q}$, where p, q are polynomials, may be expressed in the form:

$$\frac{p}{q} = g + \frac{r}{s},$$

where g is a polynomial, and $\frac{r}{s}$ is a proper rational function.

We have seen that:

$$\frac{1}{(x-1)(x+1)} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

More generally:
Let $\frac{r}{s}$ be a proper rational function.

Factor s as a product of powers of distinct irreducible factors:

$$s = \cdots (x-a)^m \cdots (\underbrace{x^2 + bx + c}_{\substack{\text{irreducible} \\ \text{i.e. } b^2 - 4c < 0}})^n \cdots.$$

Then:

Fact.

The proper rational function $\frac{r}{s}$ may be written as a sum of rational functions as follows:

$$\frac{r}{s} = \cdots + \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m} + \cdots + \frac{B_1 x + C_1}{x^2+bx+c} + \frac{B_2 x + C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_n x + C_n}{(x^2+bx+c)^n} + \cdots,$$

where the A_i, B_i, C_i are constants.

Example.

- $\displaystyle\int \frac{x^2 - x - 2}{2x^3 + 3x^2 - 2x}\,dx$
- $\displaystyle\int \frac{x}{(x^2+4)(x-3)}\,dx$
- $\displaystyle\int \frac{x^5}{x^4 - 1}\,dx$
- $\displaystyle\int \frac{8x^2}{x^4 + 4}\,dx$
