Sequences

A **sequence** is an ordered list of numbers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Common notations:

$$\{a_n\},\ \{a_n\}_{n\in\mathbb{N}},\ \{a_n\}_{n=1}^{\infty}$$

Examples

- $$a_n = \sqrt{n}\,, \quad n \in \mathbb{N}$$
  $$\{a_n\}_{n\in\mathbb{N}} = \{1, \sqrt{2}, \sqrt{3}, \ldots\}.$$

- $$b_n = (-1)^{n+1}\frac{1}{n}, \quad n \in \mathbb{N}$$
  $$\{b_n\} = \left\{1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots\right\}.$$

- (Fibonacci Sequence)
  $$a_1 = 1, a_2 = 1$$
  $$a_n = a_{n-2} + a_{n-1} \text{ for } n \geq 3.$$
  $$\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, \ldots\}$$
  In this case we say that the sequence $\{a_n\}$ is defined **recursively**.

Sometimes, the terms a_n of a sequence approach a single value L as n tends to infinity.

> **Definition.** We say that the **limit** of a sequence $\{a_n\}$ is equal to L if for all real numbers $\varepsilon > 0$ the exists a number $N > 0$ such that $|a_n - L| < \varepsilon$ for all $n > N$.

If such a number L exists, we say that:

> $\{a_n\}$ converges to L,

and write:

$$\lim_{n\to\infty} a_n = L.$$

If no such L exists, we say that $\{a_n\}$ diverges.
If the values of a_n increase (resp. decrease) without bound, we say that $\{a_n\}$ diverges to ∞ (resp. $-\infty$), and write:

$$\lim_{n\to\infty} a_n = \infty \quad (\text{resp. } -\infty).$$

Some helpful results:

- Constant sequence: If $a_n = c$ for all n, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c = c$.
- Sum/Difference rule: If both $\{a_n\}$ and $\{b_n\}$ converge, then:

$$\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n.$$

- Product Rule: If both $\{a_n\}$ and $\{b_n\}$ converge, then:

$$\lim_{n\to\infty} a_n b_n = \left(\lim_{n\to\infty} a_n\right) \cdot \left(\lim_{n\to\infty} b_n\right).$$

- Quotient Rule: If both $\{a_n\}$ and $\{b_n\}$ converge, and $\lim_{n\to\infty} b_n \neq 0$, then:

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}.$$

- $$\lim_{n\to\infty} \frac{1}{n} = 0.$$

- In general, if $\lim_{n\to\infty} a_n = +\infty$ or $\lim_{n\to\infty} a_n = -\infty$, we have:

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

Examples

- $\lim\limits_{n\to\infty} \dfrac{3n^2 - 2n + 7}{2n^2 + 3}$

- $\lim\limits_{n\to\infty} \dfrac{-3n^2}{\sqrt[3]{27n^6 - 5n + 1}}$

- $\lim\limits_{n\to\infty} \sqrt{4n^2 + n} - \sqrt{4n^2 - 1}$

A sequence $\{a_n\}$ is said to be **increasing** if $a_{n+1} > a_n$ for all n, and **decreasing** if $a_{n+1} < a_n$ for all n.

Monotone Convergence Theorem. If $\{a_n\}$ is either:

> *nondecreasing* (i.e. $a_{n+1} \geq a_n$ for all n) and bounded *above* (i.e. There exists a number M such that $a_n < M$ for all n.),

or

> *nonincreasing* (i.e. $a_{n+1} \leq a_n$ for all n) and bounded *below* (i.e. There exists a number M such that $a_n > M$ for all n.),

then $\{a_n\}$ converges.

Moreover,

if $\{a_n\}$ is nondecreasing and $a_n < M$ for all n, then $\lim_{n\to\infty} a_n \leq M$.

If $\{a_n\}$ is nonincreasing and $a_n > M$ for all n, then $\lim_{n\to\infty} a_n \geq M$.

Example. Let $\{a_n\}$ be a sequence of positive real numbers, which is defined by

$$a_1 = 1 \qquad \text{and} \qquad a_n = \frac{12a_{n-1} + 12}{a_{n-1} + 13} \text{ for } n > 1.$$

1. Prove that $a_n \leq 3$.
2. Prove that $\{a_n\}$ converges (i.e. $\lim\limits_{n\to\infty} a_n$ exists), and find its limit.

The Sandwich Theorem for Sequences. Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences such that:

$$a_n \leq b_n \leq c_n$$

for all n sufficiently large. If

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L,$$

then $\lim_{n\to\infty} b_n = L$ also.

Examples.

1. Find the following limit: $\lim\limits_{n\to\infty} \dfrac{\sin(2^n) + (-1)^n \cos(2^n)}{n^3}$.

2.
   - Prove that $\dfrac{2^n}{n!} \leq \dfrac{4}{n}$ for all natural numbers $n \geq 2$.
   - Then, show that $\lim\limits_{n\to\infty} \dfrac{2^n}{n!} = 0$.

3. Suppose $0 < a < 1$. Let $b = \frac{1}{a} - 1$. For $n \geq 2$, use the binomial theorem to show that

$$\frac{1}{a^n} \geq \frac{n(n-1)}{2} b^2.$$

Then, show that:

$$\lim_{n \to \infty} n a^n = 0.$$