THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2016–2017)
Introduction to Topology
Pre-Notes 01 Metric

We choose to start the study of topology from a natural extension of absolute value or modulus between two numbers, that is, a distance measurement on a set. This provides an easy intuition of the study.

The concept of metric space is trivially motivated by the easiest example, the Euclidean space. Namely, the metric space $(\mathbb{R}^n, d)$ with

$$d(x,y) = \|x-y\| = \left[\sum_{k=1}^{n}(x_k - y_k)^2\right]^{1/2},$$

where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. This is usually referred to as the *standard metric* on $\mathbb{R}^n$.

There are other metrics on $\mathbb{R}^n$, customarily called ℓ_p-metric, for $p \geq 1$, where

$$d_p(x,y) = \|x-y\|_p = \left[\sum_{k=1}^{n}(x_k - y_k)^p\right]^{1/p}.$$

In this sense, the standard metric is actually the ℓ_2-metric. There is also the ℓ_∞-metric given by

$$d_\infty(x,y) = \max\left\{|x_k - y_k| : k = 1, \ldots, n\right\}.$$

Why can we say that ℓ_p and ℓ_∞ are also *valid* ways to measure distance? **What** are the essential properties?

Let X be a nonempty set. A *metric* on X is a function $d\colon\ X \times X \to [0, \infty)$, that is, $d(x,y) \geq 0$, satisfying the followings

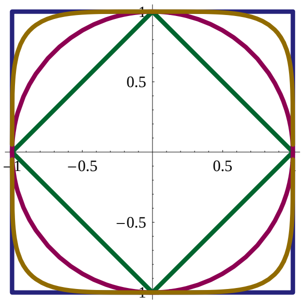
- $d(x,y) = 0$ if and only if $x = y$;
- $d(x,y) = d(y,x)$ for all $x, y \in X$;
- $d(x,y) + d(y,z) \geq d(x,z)$ for all $x, y, z \in X$.

The pair (X, d) is called a *metric space*.

The first criterion emphasizes that a zero distance is exactly equivalent to being the same point. The second symmetry criterion is natural. The third criterion is usually referred to as the *triangle inequality*. It turns out in most examples, the triangle inequality is the crucial one.

Try to verify the properties of metric in the cases of ℓ_p and ℓ_∞. Note that the triangle inequality is satisfied by the ℓ_p-metric on $\mathbb{R}^n$ for all $p \geq 1$ and $p = \infty$. There is something wrong when $0 < p < 1$.

Let us understand the ℓ_p-metric for different values of p by considering the pictures of the sets of *radius* 1, i.e., $\{ x \in \mathbb{R}^n : d_p(x, 0) = 1 \}$. The pictures of other radii are similar and they are convex when $p \geq 1$.



The pictures for $p = 1$ (green), $p = 2$ (purple), $p = 5$ (brown), and $p = \infty$ (blue).

Try to consider both analytically and geometrically why $0 < p < 1$ will not give a metric.

The *discrete metric* on any nonempty set X is defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

That this defines a metric can be readily proved by verifying the criteria case by case. The discrete metric is kind of an uninteresting metric because any two distinct points will have a fixed distance afar. However, it often serves as an example to check certain property of a space.

Similarly, we can understand the discrete metric by **doing** the following exercise about sets of a certain radius. This is often a good way to understand a metric.

Let (X, d) be the discrete metric space and $x_0 \in X$. Determine the sets $\{ x \in X : d(x, x_0) < r \}$ for different values of $r > 0$.

Metric can be defined on space of functions, although later you may find that it is sometimes not so successful. Similar to the situation of $\mathbb{R}^n$, there are several metrics on a function space. For simplicity, let $X = \mathcal{C}([a, b], \mathbb{R})$ be the set of all continuous real valued functions defined on an interval $[a, b]$. We have metrics d_p for $p \geq 1$ and $p = \infty$, namely, for $f, g \in X$,

$$d_p(f, g) = \left[\int_a^b |f(t) - g(t)|^p \, dt \right]^{1/p},$$

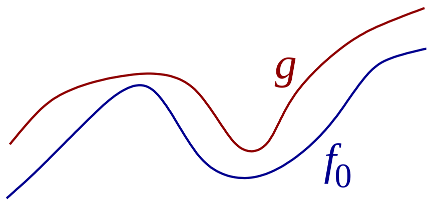
$$d_\infty(f, g) = \sup \{ |f(t) - g(t)| : t \in [a, b] \} .$$

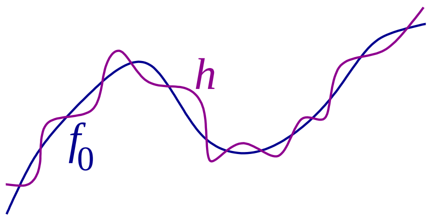
The proof for that these are metrics is similar to the Euclidean cases. In fact, d_∞ is a metric on $\mathcal{B}([a, b], \mathbb{R})$, the set of bounded functions on $[a, b]$. However, there is difficulty extending d_p to a larger set of functions, even the functions are integrable.

A suitable choice of metric may have the effect of good comparison. Let X be the set of all continuously differentiable (C^1) functions on an interval $[a, b]$ and

$$d(f, g) = \sup \{ |f(t) - g(t)| : t \in [a, b] \} + \sup \{ \left| f'(t) - g'(t) \right| : t \in [a, b] \} .$$

With this choice of metric, for the functions illustrated below, $d(f,g) < d(f,h)$ because the contribution of derivatives $|f'(t) - h'(t)|$ is large.





To finish, we will give two examples. Both examples are related to error-correcting applications. You are encouraged to understand them by the two standard tasks: verify the metric conditions and think of the typical situation of $d(x, x_0) < r$.

Let $X = \{0, 1\}^n$, i.e., it contains points of n-coordinates of 0 and 1; $d(x, y)$ is the number of different coordinates between x and y.

Let X be the set of finite sequences of alphabets. For example, "homomorphic", "homeomorphic", "holomorphic", "homotopic", "homologous" are elements of X. Suppose there are three valid operations, inserting an alphabet, deleting an alphabet, and replacing an alphabet by another. For two elements $x, y \in X$, define $d_3(x, y)$ by the minimum number of operations required to transform x to y.

We may also consider replacing an alphabet equivalent to deleting then inserting. In this case, we only accept two types of operations and let $d_2(x, y)$ be the minimum number of such operations.