MATH5012 Real Analysis II

Exercise 1

1. Let μ be a Radon measure in $\mathbb{R}^n$ and A be μ-measurable. Show that $\nu(E) = \mu(E \cap A)$ is a Radon measure.

2. Give an example showing that the condition on the uniform bound on the diameter cannot be removed in Vitali's covering theorem.

3. Show that for every non-empty open set G in $\mathbb{R}^n$, there is a countable, pairwise disjoint open balls B_k's in G satisfying

$$\mathcal{L}^n\left(G \setminus \bigcup_k B_k\right) = 0 \ .$$

4. Show that there is no countable, pairwise disjoint open balls whose union is the open square $(0,1)^2 \subset \mathbb{R}^2$. It is known that every open set in $\mathbb{R}^1$ can be decomposed as the union of countably many disjoint open intervals. This example shows that such property no longer holds in higher dimensions.

5. Give a proof of Lemma 6.5 when $\mu = \mathcal{L}^n$ and $\nu << \mathcal{L}^n$ using Corollary 6.2.

6. Let f be a Lebesgue measurable function in $\mathbb{R}^n$. Let $\mathcal{B}_x$ be the collection of all non-degenerate, closed balls touching x. Define the **maximal function** of f by

$$(Mf)(x) = \sup_{\overline{B} \in \mathcal{B}_x} \frac{1}{\mathcal{L}^n(\overline{B})} \int_{\overline{B}} |f(y)| \ dy.$$

   Show that

   (a) $\{x : (Mf)(x) > \alpha\}$ is open, $\forall \alpha \in [0, \infty)$.

   (b) $M(f+g) \leq Mf + Mg$.

7. For $f \in L^1(\mathbb{R}^n)$, establish the following facts:

(a)

$$\mathcal{L}^n \{x : Mf(x) > \alpha\} \le \frac{C(n)\,\|f\|_{L^1}}{\alpha}\ , \quad \alpha > 0\ ,$$

where $C(n)$ is a dimensional constant.

(b) Mf is finite a.e.

(c) $Mf \notin L^1(\mathbb{R}^n)$ unless $f = 0$ a.e. Hint: $Mf(x) \ge \dfrac{c}{|x|^n}$, $c > 0$, $|x| \ge 1$.

8. Consider

$$\phi(x) = \begin{cases} \dfrac{1}{|x|\left(\log\frac{1}{|x|}\right)^2}, & |x| \le \dfrac{1}{2}, \\ 0, & |x| > \dfrac{1}{2}, \ x \in \mathbb{R}. \end{cases}$$

Show that (a) $\phi \in L^1(\mathbb{R})$ and (b) $M\phi \notin L^1_{loc}(\mathbb{R})$. Suggestion: To establish

$$M\phi(x) \ge \frac{c}{|x| \log\frac{1}{|x|}},\ c > 0,\ |x| \le \frac{1}{2}.$$

Maximal functions were introduced by Hardy and Littlewood in the context of harmonic analysis. Here in Problems 6-8 we use them to illustrate the power of the covering lemmas. You may also use the version of Vitali covering lemma in [R]. Later we will show that the maximal function of an L^p-function is again an L^p-function when $p \in (0, \infty)$.