THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2022-2023 Term 1
Suggested Solutions of WeBWork Coursework 7

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1. (1 point) Suppose that

$$f(x) = 7x^2 \ln(x), \quad x > 0.$$

(A) List all the critical points of $f(x)$. Note: If there are no critical points, enter 'NONE'.

____

(B) Find all intervals (separated by commas if more than one) where $f(x)$ is increasing. Pay attention to endpoints!
Note: Use 'INF' for ∞, '-INF' for $-\infty$, and use 'U' for the union symbol. If there is no interval, enter 'NONE'.
Increasing: ____________________

(C) Find all intervals (separated by commas if more than one) where $f(x)$ is decreasing. Pay attention to endpoints!
Decreasing: ____________________

(D) List the x values of all local maxima of $f(x)$. If there are no local maxima, enter 'NONE'.
x values of local maximums = ____________

(E) List the x values of all local minima of $f(x)$. If there are no local minima, enter 'NONE'.
x values of local minimums = ____________

(F) Find all open intervals where $f(x)$ is concave up.
Concave up: ____________________

(G) Find all open intervals where $f(x)$ is concave down.
Concave down: ____________________

Solution: (A) It is easy to see that $f(x)$ is differentiable on the domain $(0, \infty)$. We compute its derivative

$$f'(x) = 14x \ln(x) + 7x^2 \cdot \frac{1}{x} = 14x \ln(x) + 7x = 7x(1 + 2\ln(x))$$

The equation $f'(x) = 7x(1 + 2\ln(x)) = 0$ has only one solution $x = e^{-\frac{1}{2}}$ on $(0, \infty)$. So $f(x)$ has only one critical point, which is at $x = e^{-\frac{1}{2}}$.

(B) On the domain $(0, \infty)$, $7x > 0$. So $f'(x) > 0$ if and only if $1 + 2\ln(x) > 0$, which is on $(e^{-\frac{1}{2}}, \infty)$. Since $f(x)$ is continuous at $x = e^{-\frac{1}{2}}$, $f(x)$ is increasing on $[e^{-\frac{1}{2}}, \infty)$.

(C) Similar to the last question, $f(x) < 0$ when $1 + 2\ln(x) < 0$, which is $(0, e^{-\frac{1}{2}})$. Also, $f(x)$ is continuous at $x = e^{-\frac{1}{2}}$, so $f(x)$ is decreasing on $(0, e^{-\frac{1}{2}}]$.

(D & E) It suffices to check the behavior of f around the only critical point. As the function is decreasing on the left of the critical point and increasing on the right, $x = e^{-\frac{1}{2}}$ is the only local minima and f has no local maxima.

(F) We compute the second derivative

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}(14x\ln(x) + 7x) = 14\left(\ln(x) + x \cdot \frac{1}{x}\right) + 7 = 7(2\ln(x) + 3)$$

Since $f''(x) > 0$ only when $x > e^{-\frac{3}{2}}$, $f(x)$ is concave up on $(e^{-\frac{3}{2}}, \infty)$.

(G) Similar to the last question, $f''(x) < 0$ only when $0 < x < e^{-\frac{3}{2}}$, so $f(x)$ is concave down on $(0, e^{-\frac{3}{2}})$.

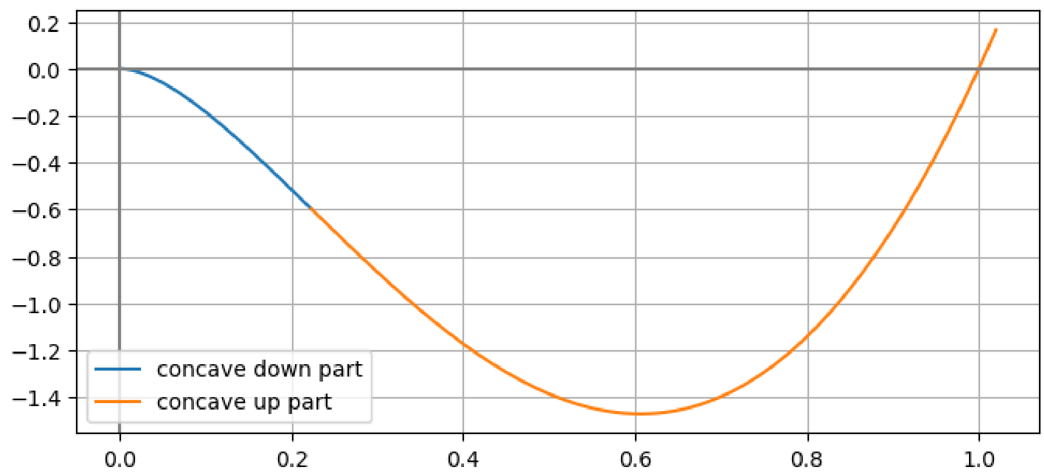
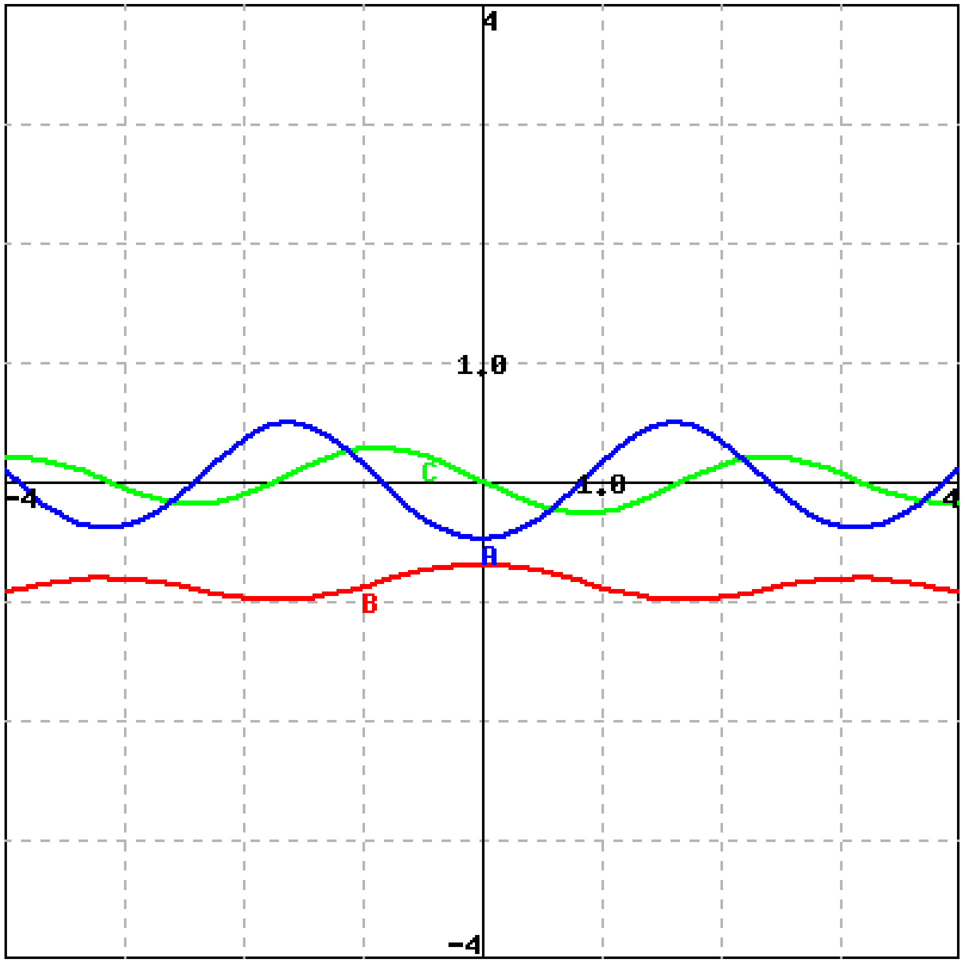


Figure 1: The graph of $f(x) = 7x^2\ln(x)$



2. (1 point) Identify the graphs A (blue), B(red) and C (green) as the graphs of a function and its derivatives:

___ is the graph of the function

___ is the graph of the function's first derivative

___ is the graph of the function's second derivative

Solution: Call the functions in blue, red, and green f_B, f_R, f_G respectively.

At $x = 0$, only f_G takes the value 0, and the graphs of both f_B and f_R are flat near $x = 0$. So either $f'_G = f_B$ or $f'_G = f_R$.

Near $x = 1$, f_G becomes increasing from decreasing, while f_R stays negative. So we cannot have $f'_G = f_R$. Hence $f'_G = f_B$.

So either $f'_R = f_G$ or $f_B = f'_R$. Near $x = 0$, f_R becomes decreasing from increasing, while f_B stays negative. Thus we cannot have the latter case, and conclude that $f'_R = f_G$.

Therefore f_B is the original function, f_G is the first derivative, and f_R is the second derivative.

3. (1 point) Find the maximum area of a triangle formed in the first quadrant by the x-axis, y-axis and a tangent line to the graph of $f = (x+7)^{-2}$.

   Area =_________

   Solution:
   Let $P\left(t, \frac{1}{(t+7)^2}\right)$ be a point on the graph of the curve $y = \frac{1}{(x+7)^2}$ in the first quadrant. The tangent line to the curve at P is

   $$L(x) = \frac{1}{(t+7)^2} - \frac{2(x-t)}{(t+7)^3},$$

   which has x-intercept $a = \frac{3t+7}{2}$ and y-intercept $b = \frac{3t+7}{(t+7)^3}$. The area of the triangle in question is

   $$A(t) = \frac{1}{2}ab = \frac{(3t+7)^2}{4(t+7)^3}.$$

   Solve

   $$A'(t) = \frac{(3t+7)(3\cdot 7 - 3t)}{4(t+7)^4} = 0$$

   for $0 \le t$ to obtain $t = 7$ is the only critical point on $(0, \infty)$. To see the maximum of A, we only need to compare the values of boundary points with the critical value. Because $A(0) = \frac{1}{4\cdot 7}$, $A(7) = \frac{1}{2\cdot 7}$ and $A(t) \to 0$ as $t \to \infty$, it follows that the maximum area is $A(7) = 0.0714286$.

4. (1 point) If 1100 square centimeters of material is available to make a box with a square base and an open top, find the largest possible volume of the box.

   Volume = _____________ (include **units**)

   Solution:

   To solve this problem, we will need to write a formula for the volume of the box in terms of one of its dimensions, and then use derivatives to find the dimensions at which the box has a maximum volume. Let x be the length of the sides of the square base. Then, if h is the height of the box, the volume is given by x^2h. We need to find an expression for the height h in terms of x.

   This is where we use our information about the amount of material used in constructing the box. If the base of the box has sides of length x, then x^2 square centimeters of material are used to make the base. Therefore, we have only $1000 - x^2$ square centimeters of material left to make the sides, of which there are four. Each of the sides uses hx square centimeters of material. Therefore, we get the formula:

$$1100 - x^2 = 4(hx) \Rightarrow h = \frac{1100 - x^2}{4x}$$

Plugging this into our formula for volume, we can now write out $v(x)$ as:

$$v(x) = x^2\left(\frac{1100 - x^2}{4x}\right) = \frac{1100x - x^3}{4}$$

Now, we take the derivative of this expression, using the rules for taking derivatives of polynomials, to get $v'(x) = \frac{1100}{4} - \frac{3}{4}x^2$. Setting this equal to 0 will give us the critical points. When solving, remember that this is a real world situation, so we can not have a negative value for x (which is a length).

$$\begin{aligned} v'(x) &= 0 \\ \tfrac{1100}{4} - \tfrac{3}{4}x^2 &= 0 \\ \tfrac{3}{4}x^2 &= \tfrac{1100}{4} \\ x^2 &= \tfrac{1100}{3} \\ x &= \sqrt{\tfrac{1100}{3}} \end{aligned}$$

Now, plugging this width into our formula for volume, $v(x)$, we get the maximal volume of $v(\sqrt{\frac{1100}{3}}) \approx 3510.57$ cm^3.

5. (1 point) Use L'Hôpital's Rule (possibly more than once) to evaluate the following limit $\lim_{x\to\infty}(\frac{8x^3+4x^2}{3x^3-5}) =$ __________
If the answer equals ∞ or $-\infty$, write INF or -INF in the blank.

Solution:

$$\begin{aligned} \lim_{x\to\infty}\frac{8x^3+4x^2}{3x^3-5} &= \lim_{x\to\infty}\frac{(8x^3+4x^2)'}{(3x^3-5)'} = \lim_{x\to\infty}\frac{24x^2+8x}{9x^2} \\ &= \lim_{x\to\infty}\frac{(24x^2+8x)'}{(9x^2)'} = \lim_{x\to\infty}\frac{48x+8}{18x} \\ &= \lim_{x\to\infty}\frac{(48x+8)'}{(18x)'} = \lim_{x\to\infty}\frac{48}{18} \\ &= \frac{8}{3} \end{aligned}$$

6. (1 point) Compute

$\lim\limits_{x \longrightarrow 0} \dfrac{e^x - e^{-x}}{2\sin x} =$ ______________

Solution:

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{2\sin x} = \lim_{x\to 0} \frac{(e^x - e^{-x})'}{(2\sin x)'} = \lim_{x\to 0} \frac{e^x + e^{-x}}{2\cos x} = \frac{e^0 + e^0}{2\cos 0} = 1$$

7. (1 point) Let $f(x) = \dfrac{\ln x}{1 + (\ln x)^2}$ for x in $(0, \infty)$. Find

a) $\lim\limits_{x\to 0^+} f(x) =$ __________

b) $\lim\limits_{x\to\infty} f(x) =$ __________

Solution:

a)

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{\ln x}{1 + (\ln x)^2} = \lim_{x\to 0^+} \frac{(\ln x)'}{(1 + (\ln x)^2)'} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{2\ln x \cdot \frac{1}{x}} = \lim_{x\to 0^+} \frac{1}{2\ln x} = 0$$

b)

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\ln x}{1 + (\ln x)^2} = \lim_{x\to\infty} \frac{(\ln x)'}{(1 + (\ln x)^2)'} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2\ln x \cdot \frac{1}{x}} = \lim_{x\to\infty} \frac{1}{2\ln x} = 0$$

8. (1 point) Evaluate the following limit
$\lim\limits_{x\to 0}(\cot(7x) - \frac{1}{7x}) =$ __________
If the answer equals ∞ or $-\infty$, write INF or -INF in the blank.

Solution:

$$\begin{aligned}\lim_{x\to 0}\left(\cot(7x)-\frac{1}{7x}\right) &= \lim_{7x\to 0}\left(\cot(7x)-\frac{1}{7x}\right) = \lim_{t\to 0}\left(\cot(t)-\frac{1}{t}\right)\\ &= \lim_{t\to 0}\frac{t\cos(t)-\sin(t)}{t\sin(t)} = \lim_{t\to 0}\frac{(t\cos(t)-\sin(t))'}{(t\sin(t))'}\\ &= \lim_{t\to 0}\frac{-t\sin(t)}{\sin(t)+t\cos(t)} = -\lim_{t\to 0}\frac{(t\sin(t))'}{(\sin(t)+t\cos(t))'}\\ &= \lim_{t\to 0}\frac{\sin(t)+t\cos(t)}{2\cos(t)-t\sin(t)}\\ &= \frac{\sin(0)+0\cdot\cos(0)}{2\cos(0)-0\cdot\sin(0)}\\ &= 0\end{aligned}$$

9. (1 point) Evaluate the following limit:

$$\lim_{x\to\infty}\left(1+\frac{6}{x}\right)^{\frac{x}{9}}$$

____

Solution: Note that

$$\ln\left(1+\frac{6}{x}\right)^{\frac{x}{9}} = \frac{x}{9}\ln\left(1+\frac{6}{x}\right),$$

so

$$\left(1+\frac{6}{x}\right)^{\frac{x}{9}} = e^{\frac{x}{9}\ln\left(1+\frac{6}{x}\right)}.$$

We first consider the limit $\lim_{x\to\infty}\frac{x}{9}\ln\left(1+\frac{6}{x}\right)$,

$$\begin{aligned}\lim_{x\to\infty}\frac{x}{9}\ln\left(1+\frac{6}{x}\right) &= \lim_{x\to\infty}\frac{\ln\left(1+\frac{6}{x}\right)}{\frac{9}{x}} = \lim_{x\to\infty}\frac{(\ln\left(1+\frac{6}{x}\right))'}{(\frac{9}{x})'}\\ &= \lim_{x\to\infty}\frac{\frac{x}{x+6}\cdot(-6x^{-2})}{-9x^{-2}} = \frac{2}{3}\lim_{x\to\infty}\frac{x}{x+6}\\ &= \frac{2}{3}.\end{aligned}$$

So

$$\lim_{x\to\infty}\left(1+\frac{6}{x}\right)^{\frac{x}{9}} = \lim_{x\to\infty}e^{\frac{x}{9}\ln\left(1+\frac{6}{x}\right)} = e^{\lim_{x\to\infty}\frac{x}{9}\ln\left(1+\frac{6}{x}\right)} = e^{\frac{2}{3}}.$$

Alternatively, this problem can be solved without using L'Hôpital's Rule:

Solution:

$$\lim_{x\to\infty}\left(1+\frac{6}{x}\right)^{\frac{x}{9}} = \lim_{x\to\infty}\left[\left(1+\frac{6}{x}\right)^{\frac{x}{6}}\right]^{\frac{2}{3}}$$
$$= \left[\lim_{\frac{x}{6}\to\infty}\left(1+\frac{6}{x}\right)^{\frac{x}{6}}\right]^{\frac{2}{3}} = \left[\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t}\right]^{\frac{2}{3}}$$
$$= e^{\frac{2}{3}},$$

where the well-known limit

$$\lim_{t\to\infty}\left(1+\frac{1}{t}\right)^{t} = e$$

is applied.

10. (1 point) Evaluate the following limit.

$\lim_{x\to 0+}(\csc x)^{\tan x} =$ __________

Solution: We first consider the limit $\lim_{x\to 0+}\tan x\ln\csc x$

$$\lim_{x\to 0+}\tan x\ln\csc x = \lim_{x\to 0+}\frac{\ln\csc x}{\frac{1}{\tan x}} = \lim_{x\to 0+}\frac{(\ln\csc x)'}{(\frac{1}{\tan x})'}$$
$$= \lim_{x\to 0+}\frac{\frac{1}{\csc x}\cdot\frac{-\cos x}{\sin^2 x}}{-\frac{1}{\sin^2 x}} = \lim_{x\to 0+}\sin x\cos x$$
$$= 0.$$

So

$$\lim_{x\to 0+}(\csc x)^{\tan x} = \lim_{x\to 0+}e^{\tan x\ln\csc x} = e^{\lim_{x\to 0+}\tan x\ln\csc x} = e^0 = 1.$$

11. (1 point) Compute

$\lim_{x\longrightarrow 0}(\cos x)^{1/x^2} =$ ______________

Solution: The limit is of the form 1^∞. Let
$y = (\cos x)^{1/x^2}$ and hence $\ln y = \dfrac{1}{x^2}\ln(\cos x)$.

We obtain:

$$\lim_{x \longrightarrow 0} \frac{1}{x^2} \ln(\cos x) = \lim_{x \longrightarrow 0} \frac{\ln(\cos x)}{x^2}.$$

This limit is of the form $\frac{0}{0}$, so we apply the Rule of L'Hôpital twice:

$$\lim_{x \longrightarrow 0} \frac{\ln(\cos x)}{x^2} = \lim_{x \longrightarrow 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x} = \lim_{x \longrightarrow 0} \frac{-\tan x}{2x} =$$

$$= \lim_{x \longrightarrow 0} \frac{-\sec^2 x}{2} = \frac{-1}{2}.$$

So $\lim_{x \longrightarrow 0} \ln y = -1/2$ and

$$\lim_{x \longrightarrow 0} y = \lim_{x \longrightarrow 0} (\cos x)^{1/x^2} = e^{-1/2} = \frac{1}{\sqrt{e}}.$$

12. (1 point) Find the second-degree Taylor polynomial for $f(x) = 2x^2 - 6x + 6$ about $x = 0$.
$P_2(x) =$ ____________________

What do you notice about your polynomial?

Solution:

We note that $f(0) = 6$; $f'(x) = 4x - 6$, so that $f'(0) = -6$; and $f''(x) = 4$, so that $f''(0) = 4$.

Thus

$$P_2(x) = 6 - 6x + \frac{4}{2!}x^2 = 6 - 6x + 2x^2.$$

We notice that $f(x) = P_2(x)$ in this case, which makes sense because $f(x)$ is a polynomial.

13. (1 point) Find the first four Taylor polynomials about $x = x_0$.
$\ln(x + 7); x_0 = -6$

Solution: $f(x) = \ln(x + 7)$,
$f(-6) = \ln(1) = 0$;

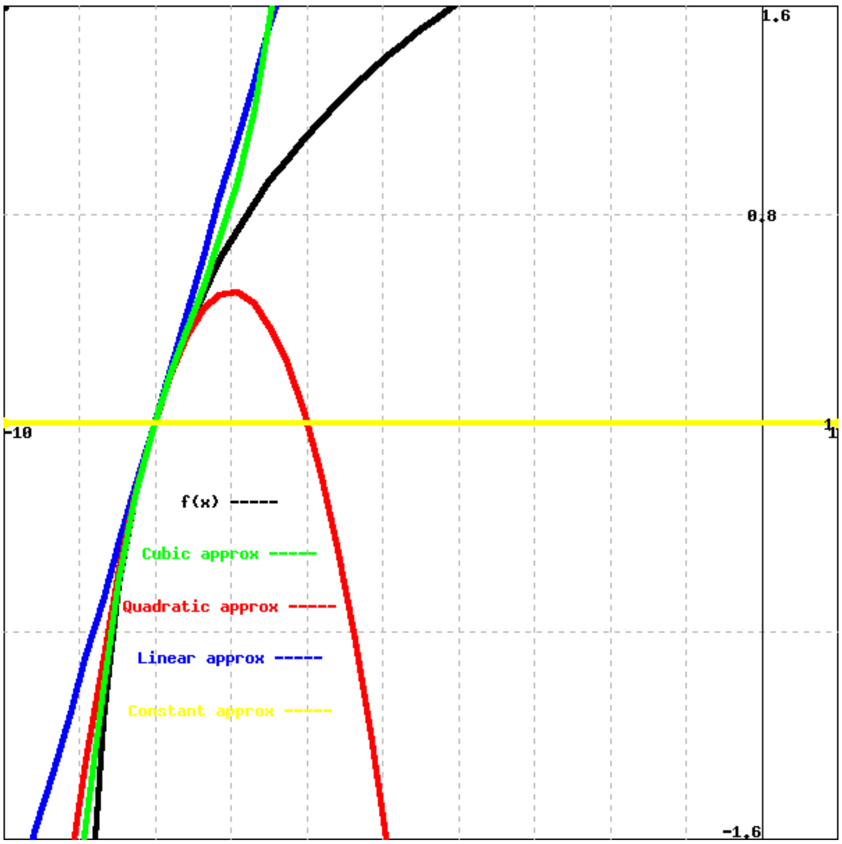
$f'(x) = \frac{1}{x+7}$,
$f'(-6) = 1$;
$f''(x) = -\frac{1}{(x+7)^2}$,
$f''(-6) = -1$;
$f^{(3)}(x) = \frac{2}{(x+7)^3}$,
$f^{(3)}(-6) = 2$.
Then we have

$$P_3(x) = 0 + (x+6) + \frac{-1}{2!}(x+6)^2 + \frac{2}{3!}(x+6)^3$$
$$= (x+6) - \frac{1}{2}(x+6)^2 + \frac{1}{3}(x+6)^3,$$

$$P_2(x) = (x+6) - \frac{1}{2}(x+6)^2,$$

$$P_1(x) = x + 6,$$

$$P_0(x) = 0.$$



14. (1 point) Consider the function $f(x) = \sqrt{x+1}$.
Let T_n be the n^{th} degree Taylor approximation of $f(10)$ about $x = 8$.

Find:

$T_1 =$ _____

$T_2 =$ _____

$T_3 =$ _____

Use 3 decimal places in your answer

Solution: In general the n^{th} degree approximation of $f(10)$ about $x = 8$ is given by:
$T_n = f(8) + f'(8)(10-8) + \ldots + \frac{f^{(n)}(8)}{n!}(10-8)^n$
$f'(x) = \frac{1}{2\sqrt{1+x}}$,
$f''(x) = -\frac{1}{4(1+x)^{\frac{3}{2}}}$,
$f'''(x) = \frac{3}{8(1+x)^{\frac{5}{2}}}$.

Therefore,
$T_1 = \frac{10}{3} \approx 3.333$,
$T_2 = \frac{179}{54} \approx 3.315$,
$T_3 = \frac{806}{243} \approx 3.317$.

15. (1 point) **Taylor and Maclaurin Series**: Compute the Taylor Series below.
$e^x =$__+__x+__x^2+__x^3+__x^4+...
$\cos^2 x =$__+__x+__x^2+__x^3+__x^4+...
$x^x =$__+__$(x-1)$+__$(x-1)^2$+__$(x-1)^3$+__$(x-1)^4$+...
$4x^4 + 3x^3 + 2x^2 + x + 1 =$__+__$(x-1)$+__$(x-1)^2$+__$(x-1)^3$+__$(x-1)^4$

Solution: For convenience, denote $f(x) = e^x$, $g(x) = \cos^2 x$, $h(x) = x^x$ and $k(x) = 4x^4 + 3x^3 + 2x^2 + x + 1$.
$f^{(n)}(x) = e^x$ for all n. Therefore, $f^{(n)}(0) = 1$ for all n. Hence, we have

$$\begin{aligned} e^x &= 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + ... \\ &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + ... \end{aligned}$$

Write $g(x) = \frac{1}{2}(\cos(2x) + 1)$. Then $g'(x) = -\sin(2x), g''(x) = -2\cos(2x), g^{(3)}(x) = 4\sin(2x), g^{(4)}(x) = 8\cos(2x)$. Therefore, $g(0) = 1, g'(0) = 0, g''(0) = -2, g^{(3)}(0) = 0, g^{(4)}(0) = 8$. Then we have

$$\begin{aligned} \cos^2 x &= 1 + \frac{1}{2!}(-2)x^2 + \frac{1}{4!}(8)x^4 + ... \\ &= 1 - x^2 + \frac{1}{3}x^4 + ... \end{aligned}$$

Write $h(x) = e^{x \ln x}$. Then $h'(x) = x^x(\ln x + 1), h''(x) = x^{(x-1)}(x + x(\ln x)^2 + 2x\ln x + 1), h^{(3)}(x) = x^{(x-2)}(x^2 + x^2(\ln x)^3 + 3x^2(\ln x)^2 + 3x + 3x(x+1)\ln x + 1), h^{(4)}(x) = x^{(x-3)}(x^3 + x^3(\ln x)^4 + 4x^3(\ln x)^3 + 6x^2 + 6x^2(x+1)(\ln x)^2 + 4x(x^2+3x-1)\ln x - x + 2$.
Therefore, $h(1) = 1, h'(1) = 1, h''(1) = 2, h^{(3)}(1) = 3, h^{(4)}(1) = 8$. Then we have

$$\begin{aligned} x^x &= 1 + (x-1) + \frac{1}{2!}(2)(x-1)^2 + \frac{1}{3!}(3)(x-1)^3 + \frac{1}{4!}(8)(x-1)^4 + ... \\ &= 1 + (x-1) + (x-1)^2 + \frac{1}{2}(x-1)^3 + \frac{1}{3}(x-1)^4 + ... \end{aligned}$$

$k'(x) = 16x^3 + 9x^2 + 4x + 1, k''(x) = 48x^2 + 18x + 4, k^{(3)}(x) = 96x + 18, k^{(4)}(x) = 96$.
Therefore, $k(1) = 11, k'(1) = 30, k''(1) = 70, k^{(3)}(1) = 114, k^{(4)}(1) = 96$. Then we

have

$$\begin{aligned}
&4x^4 + 3x^3 + 2x^2 + x + 1 \\
&= 11 + 30(x-1) + \frac{1}{2!}(70)(x-1)^2 + \frac{1}{3!}(114)(x-1)^3 + \frac{1}{4!}(96)(x-1)^4 \\
&= 11 + 30(x-1) + 35(x-1)^2 + 19(x-1)^3 + 4(x-1)^4
\end{aligned}$$

16. (1 point) Find the first four terms of the Taylor series for the function $\frac{2}{x}$ about the point $a = 2$.

$\frac{2}{x} =$ ______ + ______ + ______ + ______ + ...

Solution: The function $\frac{2}{x}$ and its first three derivatives are $f(x) = \frac{2}{x}$, $f'(x) = -\frac{2}{x^2}$, $f''(x) = \frac{4}{x^3}$, and $f'''(x) = -\frac{12}{x^4}$. Thus, evaluating these at $x = 2$, we get the terms

term 0 = 1,
term 1 = $-\frac{1}{2}(x-2)$,
term 2 = $\frac{1}{4}(x-2)^2$, and
term 3 = $-\frac{1}{8}(x-2)^3$.
Thus the series is

$$\frac{5}{x} = 1 - \frac{1}{2}(x-2) + \frac{1}{4}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots .$$

17. (1 point) Find the first three **nonzero** terms of the Taylor series for the function $f(x) = \sqrt{10x - x^2}$ about the point $a = 5$.

Solution: $f(5) = 5$,
$f'(x) = \frac{10-2x}{2\sqrt{10x-x^2}} = \frac{5-x}{\sqrt{10x-x^2}}$,
$f'(5) = 0$,
$f''(x) = -\frac{25}{(10x-x^2)^{\frac{3}{2}}}$,
$f''(5) = -\frac{1}{5}$,

$f^{(3)}(x) = -\dfrac{75(x-5)}{(10x-x^2)^{\frac{5}{2}}}$,
$f^{(3)}(5) = 0$,
$f^{(4)}(x) = -\dfrac{75(4x^2-40x+125)}{(10x-x^2)^{\frac{7}{2}}}$,
$f^{(4)}(5) = -\dfrac{3}{125}$.
Then

$$\begin{aligned}\sqrt{10x-x^2} &= 5 + \frac{1}{2!}(-\frac{1}{5})(x-5)^2 + \frac{1}{4!}(-\frac{3}{125})(x-5)^4 + \cdots \\ &= 5 - \frac{1}{10}(x-5)^2 - \frac{1}{1000}(x-5)^4 + \cdots .\end{aligned}$$

18. (1 point) Evaluate

$$\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{14x^4}.$$

Limit = ____________

Solution: Apply L'Hôpital's Rule

$$\begin{aligned}\lim_{x\to 0} \frac{\cos(x) - 1 + \frac{x^2}{2}}{14x^4} &= \lim_{x\to 0} \frac{-\sin(x) + x}{56x^3} \\ &= \lim_{x\to 0} \frac{-\cos(x) + 1}{168x^2} \\ &= \lim_{x\to 0} \frac{\sin(x)}{336x} \\ &= \lim_{x\to 0} \frac{\cos(x)}{336} \\ &= \frac{1}{336}.\end{aligned}$$

19. (1 point) Evaluate

$$\lim_{x\to 0} \frac{\ln(1-x) + x + \frac{x^2}{2}}{9x^3}$$

Answer: ____________

Solution: Apply L'Hôpital's Rule

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1-x)+x+\frac{x^2}{2}}{9x^3} &= \lim_{x\to 0}\frac{\frac{1}{x-1}+1+x}{27x^2}\\ &= \lim_{x\to 0}\frac{-\frac{1}{(x-1)^2}+1}{54x}\\ &= \lim_{x\to 0}\frac{\frac{2}{(x-1)^3}}{54}\\ &= -\frac{1}{27}\end{aligned}$$

20. (1 point) A smokestack deposits soot on the ground with a concentration inversely proportional to the square of the distance from the stack. With two smokestacks d miles apart, the concentration of the combined deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{c}{x^2} + \frac{k}{(d-x)^2}$$

where c and k are positive constants which depend on the quantity of smoke each stack is emitting. If $k = 5c$, find the point on the line joining the stacks where the concentration of the deposit is a minimum.

x_{min} = ______________________ mi

Solution:

We want to find x such that

$$S = \frac{c}{x^2} + \frac{k}{(d-x)^2} = \frac{c}{x^2} + \frac{5c}{(d-x)^2} = c\left(\frac{1}{x^2} + \frac{5}{(d-x)^2}\right)$$

is a minimum, which is the same thing as minimizing

$$f(x) = x^{-2} + 5(d-x)^{-2}$$

since c is nonnegative.

We have

$$f'(x) = -2x^{-3} - 10(d-x)^{-3}(-1) = \frac{-2}{x^3} + \frac{10}{(d-x)^3} = \frac{-2(d-x)^3 + 10x^3}{x^3(d-x)^3}.$$

Thus we want to find x such that $-2(d-x)^3 + 10x^3 = 0$, which implies $10x^3 = 2(d-x)^3$. That's equivalent to $5x^3 = (d-x)^3$, or $\frac{d-x}{x} = 5^{1/3} \approx 1.71$.

Solving for x, we have $d - x = 1.71x$, whence $x = d/(1 + 1.71)$.

To verify that this minimizes f, we take the second derivative:

$$f''(x) = 6x^{-4} + 30(d - x)^{-4} > 0$$

for any $0 < x < d$, so by the second derivative test the concentration is minimized $d/(1 + 1.71)$ miles from the smokestack.